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THE INTERACTION OF WAVES
AND OIL SPILLS

by

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RELATED REPORTS

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Moore, Stephen F., et al. POTENTIAL BIOLOGICAL EFFECTS OF HYPOTHETICAL OIL DISCHARGES IN THE ATLANTIC COAST AND GULF OF ALASKA. MITSG 74-19; NTIS COM-74-11089/AS. Cambridge: M.I.T. Sea Grant Program, 1974. 121 pp.

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Robbins, Phillips W. STUDENT PROJECTS ON THE OXIDATION BY MARINE BACTERIA OF AROMATIC COMPOUNDS FOUND IN OIL. MITSG 71-10; NTIS COM-71-00878. Cambridge: M.I.T. Sea Grant Program, 1971. 55 pp.

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Offshore Oil Task Group. THE GEORGES BANK PETROLEUM STUDY. MITSG 73-5. Cambridge: M.I.T. Sea Grant Program, 1973. 3 volumes. $10.00.

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ABSTRACT

The combined effects of interfacial elasticity and multiple fluid layer geometries upon the first order properties of surface waves are analyzed for application to the oil spill problem. The technique of matched assymptotic expansion is used in conjunction with a small amplitude assumption regarding the wave height.

The theory of interfacial elasticity is briefly reviewed, and critical parameters are identified. Recommendations for further study of oil-water adsorption and mixed monolayer elasticities are developed.

The second order mass transport properties are discussed in light of the first order solutions. It is concluded that the enormous potential of this system for significant variation in the first order properties makes further theoretical analysis highly dependent upon the availability of suitable experimental data to direct the theoretician.

ACKNOWLEDGEMENTS

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1. INTRODUCTION

A number of authors have speculated upon the possible importance of the interaction of waves and oil spills in determining the transport of oil on the surface of the ocean. Unfortunately, the oil-water system has not been the subject of a reasonably specialized analysis, with the result that speculations in the past have hinged upon theoretical results that were strictly applicable only to fairly coarse idealizations of the oil-water system. The commonly invoked classical analyses either ignored the particular visco-elastic behavior of the oil-water interface, or else they so idealized the interfacial phenomena as to cast doubt upon the generality of the solution. This paper examines the oil-water system and lays the necessary groundwork so that we can consider the interaction of waves and oil films without recourse to idealizations whose physical significance is unclear. The method of matched assymptotic expansions is used because of the clarity offered by this technique in elucidating the various physical scales and their contribution to the overall problem.

Since the intended use of this analysis is to shed light upon the transport of oil spills upon the open ocean, the range of physical parameters is selected to coincide with our knowledge of unconstrained oil spills on the sea. For example, we know that the thickness of the oil is typically so small the irrotational modes within the oil can be neglected. Further, the water depth is presumed to be infinite. These restrictions are not necessary from the standpoint of the application of the solution technique, but they do simplify the analysis to an important degree.

Two often referred to classical analyses are those of Harrison (1908 a,b) and Dorrestein (1951 a,b). Harrison's analysis treats the case of two superposed viscous liquids of infinite extent with no surface tension (a), and one viscous liquid occupying the lower half plane with surface tension (b). From our standpoint, the most important results obtained by Harrison are those of paper (a), particularly the viscosity induced changes in the frequency parameter. Dorrestein's papers (51a) and (51b) are an amplification of some previous studies by Lamb (32). The focus of these studies is the effect of visco-elastic surface film behavior on free surface waves assuming that the overlying fluid is

of negligible density. The analytical technique is the
same as Harrison's the difference lying in the
formulation of the tangential stress condition and the
neglect of the upper fluid. Dorrestein's results are
important because they show explicity the importance
of the elasticity parameter. The reader is cautioned,
however, that Dorrestein mistakenly identified the
dilational viscosity with the more commonly measured
monolayer surface shear viscocity. Thus his estimates
of the possible importance of this parameter in the
two-dimensional wave problem are not to be trusted.

There are several recent examples of the use
of matched asymptotic expansions in viscous fluid surface
wave problems, the one most nearly like that used here
being the work of Johns (68). The idea behind this
technique is that the flow may be considered to be
irrotational over the bulk of the region of interest,
with the exception of the areas lying near fluid interfaces
or along solid surfaces. In the irrotational regions,
the fluid velocities may be related to a potential
function satisfying Laplace's equation. However, in the
boundary layers lying near fluid interfaces and along
solid surfaces, regions of strong rotational flow are
established, and in such regions the velocities are related

to a stream function. If we steady the problem through
a suitable selection of coordinate translation and wave
generator mechanism, then we can see that the boundaries
of the region in question conform to the sub-characteristics
formed by the outer irrotational solution, so we know
from Cole's discussions (1968, pages 120-162) that the
appropriate non-dimensional inner variable is related to
the reciprocal of the square root of the Reynolds number.
The two solutions are then formally joined in a region
of overlap.

Our analysis for two superposed fluids differs
slightly from Johns' in that we make use of an asymptotic
power series expansion that explicitly incorporates both
upper and lower inner expansion parameters. This has
the effect of making some of the kinematic conditions
explicit, where previous investigations such as Dore (70)
had to rely upon order of magnitude arguments to discard
unwanted terms.

Neglecting the possible influence of wave height
upon the dynamic interactions, the oil-on-water problem
is solved for the small amplitude case. Letting ε be a
characteristic non-dimensional boundary layer depth, and α
be a characteristic non-dimensional wave amplitude, the

condition investigated is characterized by $\varepsilon \gg \alpha$. This condition allows us to use cartesian coordinates and the analysis follows Johns.

The analysis is primarily concerned with an examination of the oil layer related perturbations to the first order properties of surface waves. Particular attention is paid to the wave number perturbation and the characteristics of the velocity field in the vicinity of the interface.

The second order analysis is much more uncertain due to both an absence of suitable field observations and uncertainties regarding the proper ordering technique. Some implications of the strong damping of surface waves on the second order boundary condition are developed, but more remains to be done. The conditions under which the second order flow might be amenable to a Wiener-Hopf solution are documented and the possible implications with respect to mass transport are discussed.

2. INTERFACIAL PROPERTIES

The most important departure of this paper from
the previous studies of surface waves using the matched
asymptotic technique is the inclusion of the dynamic
properties of monolayers at the interface of the two fluids.
The known behavior of various organic compounds at an oil-
water interface makes this modification imperative in any
general theory of oil spill behavior. However, so little
is known of the properties of oil spills under oceanic
conditions that large uncertainties exist as to the
appropriate range of values to be used in specifying the
monolayer parameters. The following is a brief survey
of the subject of monolayer behavior. For a more thorough
review of the hydrodynamic aspects of the problem the
reader is referred to Levich (1962) Chapters VII and XI
(particularly sections 69, 121 and 122). The absorption
problem is nicely reviewed by E. H. Lucassen – Reynders
in Progress in Surface and Membrane Science, Vol. 10
(edited by Candenhead and Danielli, 1976); and the text
by Davies and Rideal (1963) should serve as a good
introduction to the general subject of interfacial
phenomena, although the reader is cautioned that the

organization of the book is more suited to directed study programs than quick reference.

The classical studies of monolayers and surface elasticity in general began with measurements of the variation of the surface tension of mixtures of water and various organic compounds. In the course of these experiments, it was observed that when a surface active compound such as cetyl alcohol was spread upon the air-water interface, the surface tension of the system dropped. When the surfactant molecules covered only a small percentage of the surface, the change in surface tension was very small. As more and more of the substance was added, the deviation from the uncontaminated state grew until it reached a final value, at which point the entire surface was (apparently) covered with a layer of these molecules. It was observed that the nature of the surface tension's dependence upon the surface concentration of the surfactant was highly variable, influenced strongly by the molecular characteristics of the surfactants.

Following Davies and Rideal (1963), we may consider films composed of pure compounds of low to intermediate molecular weight to be either "gaseous", "liquid expended", or "condensed" according to the

behavior of the surface tension with variations in surface
concentration. Letting, A be the surface area available
to a molecule (in Angstroms squared conventionally, i.e.,
$(10^{-8}$ cm$)^2$)* and π be the difference between the
uncontaminated and contaminated surface tensions (i.e.,
$\pi = T(A=\infty) - T(A))$ then gaseous and liquid expanded films
tend to obey the following equation:

$$(\pi - \pi_s)(A - A_o) = KT \qquad (2.1)$$

(Davies and Rideal, equations 5.1 and 5.2)

where π_s = 0
A_o = 0 $\Big\}$ for gaseous films:

$\pi_s \sim -10 \dfrac{dynes}{cm}$
$A_o \sim 20 \overset{o}{A}^2$ $\Big\}$ for liquid expanded films;

K = Boltzman's constant; and,

T = Temperature (^oK).

Condensed films do not show this gradual
variation in π with A; rather, the intermolecular

---
*Use is made in some recent papers of the nanometer
squared unit which is $(10^{-7}$ cm$)^2$.

cohesive forces tend to be so high that at low surface

concentrations the surfactant organizes itself into small

"islands", and these islands do not materially affect

the surface tension of the system. It is only as the

coverage of the surface approaches 100% that these islands

come to form a reasonably contiguous film, whereupon the

surface pressure π, grows rapidly. Liquid expanded films

also form islands at low conentrations, but sufficient

numbers of molecules exist in a free gaseous state to

affect the surface tension even at low concentrations.

Large molecules, proteins and so on, tend to

obey equations of the form:

$$\Pi = \chi \frac{KT}{A_o} \left[\ell n \left(\frac{A}{A \cdot A_o} \right) + \left(\frac{\chi-1}{\chi} \right) \frac{z}{2} \ell n \left(1 - \frac{2A_o}{zA} \right) \right] \qquad (2.2)$$

(Davies and Rideal, equation 5.14)

where χ is the degree of polymerization (a parameter related

to the number of like molecular groups forming the polymer);

and z is the coordination number, which reflects the degree

of unfolding of the molecule on the surface. The parameter

z has a value of around 2 for molecules that remains highly

coiled, while for molecules that relax into long randomly organized chains, z is nearly 4.

The primary difference between oil-water and air-water interfacial monolayers is that films composed of the same surfactant tend to exhibit higher pressures on the oil-water interface than on the air-water interface. This is caused by the similarity between the molecules of the bulk phase of the oil and the long hydrocarbon chains of the surfactant. The molecules from the bulk phase interject themselves between the surfactant molecules and this tends to reduce the cohesive forces between surfactant molecules, allowing them to exert a greater pressure on the surface.

The usual wave problem involves periodic oscillations of the interface of two pure and mutually immiscible substances. In the course of one oscillation, the local surface area will alternately expand and contract, and in so doing, it will acquire and release potential energy. During the portion of the oscillation in which the surface is being stretched, kinetic energy will be converted into the energy required to form the additional surface. During the contraction phase of the motion, the surface will be diminished (locally), releasing

potential energy to be converted to kinetic energy. A
key feature of this process is that the composition of the
interface will not change (locally or globally) during
the oscillation. The molecules that are drawn from the
bulk phase to create the new area are identical to those
already at the surface. Thus the energy required to create
additional surface remains proportional to the incremental
area formed.

The wave problem we consider in this paper
differs from the problem above in that the interface is
presumed to contain not only molecules from the pure bulk
phases, but also a number of surfactants that change the
local surface tension according to the (local)
concentration of these compounds. If all the surfactants
are stuck at the interface (i.e., they are immiscible
in either of the bulk phases), then as the interface
stretches, the local concentration will decline, increasing
the area available per molecule, and, according to either
equation 2.1 or 2.2, the local pressure will decrease.
This corresponds to an increase in the local surface
tension, which implies that the energy required to form
the new surface is greater than that required to from
an equivalent incremental area in the pure fluids problem.

In the wave problem we consider here, the
surface oscillations are presumed to be small and so the
stretching motions are also small. Further, the
variation in the surface tension associated with the
stretching of the surface is assumed to be moderate.
Thus, the additional energy required to form new surface
when we consider the combined stretching and surface
tension variation is small compared to the energy required
to form the additional area at the average surface tension
value. Therefore this is not the source of the differences
between our present problem and the more familiar surface
tension problem.

The important difference springs from the effect
of local variations in surface tension upon the tangential
stress applied to the free surface. This can best be
seen with the free body diagram of Figure 2.1. Here we
consider a small element of the lower fluid adjacent to
the interface. The monolayer region is depicted as the
thin sheet at the interface. In actual fluids, the
thickness of this region is of molecular dimensions, say
$100 \overset{\text{o}}{\text{A}}$ (10^{-6} cm), at most, and so the mass of this region
is negligible. We also choose the other dimensions in
the fluid element to be so small that the element as a

NORMAL FORCE:

$$[\sigma_{nn}{}^u - \sigma_{nn}{}^\ell]ds - T_1 \sin[\theta_1 - \theta_2] - T_2 \sin[\theta_2 - \theta_3] = 0$$

TANGENTIAL FORCE:

$$[\sigma_{sn}{}^u - \sigma_{sn}{}^\ell]ds - T_1 \cos[\theta_1 - \theta_2] + T_2 \cos[\theta_2 - \theta_3] = 0$$
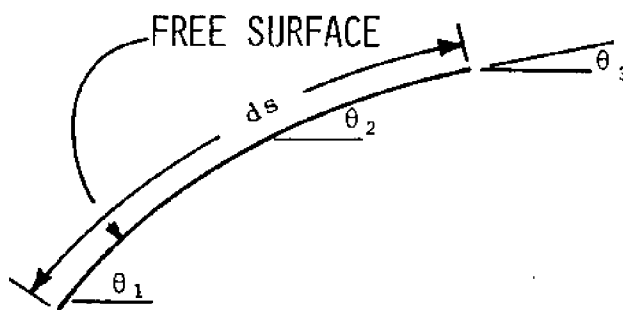
$$T_1 = T_0 - \frac{dt}{ds}\frac{ds}{2}$$

$$T_2 = T_0 + \frac{dt}{ds}\frac{ds}{2}$$

$$\theta_1 - \theta_2 = -\frac{d\theta}{ds}\frac{ds}{2}$$

$$\theta_2 - \theta_3 = -\frac{d\theta}{ds}\frac{ds}{2}$$
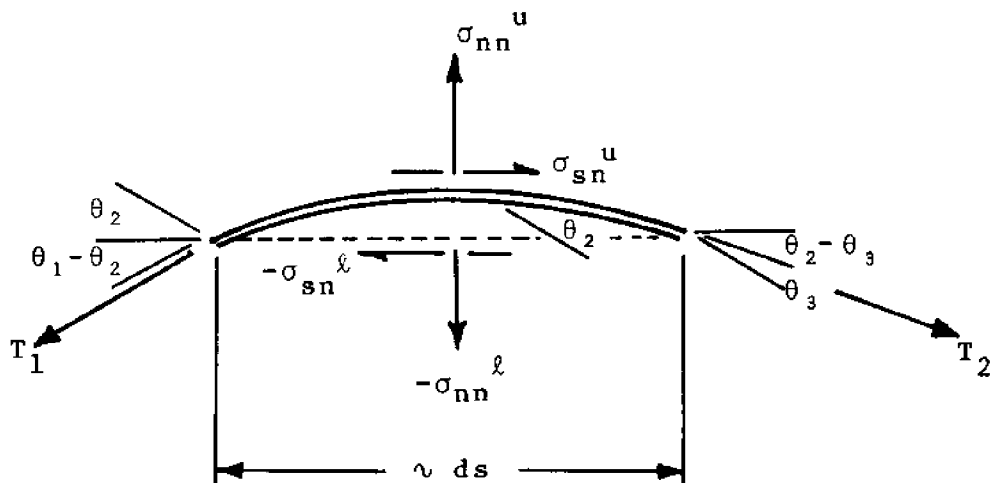
$$\sigma_{nn}{}^u - \sigma_{nn}{}^\ell = T_0 \frac{d\theta}{ds} \qquad \text{and} \qquad \sigma_{sn}{}^u - \sigma_{sn}{}^\ell = \frac{dT}{ds}$$

FIGURE 2.1

FREE BODY DIAGRAM FOR
ELASTIC INTERFACE DYNAMIC BOUNDARY CONDITION

whole is essentially without mass. Under these conditions
the sum of the stresses applied to the surfaces of the
element must be zero. Letting ds be the length of the
surface element, we find that correct to O(ds):

$$\sigma_{nn}^{\ell} - \sigma_{nn}^{u} = T \frac{d\theta}{ds} \qquad (2.3)$$

$$\sigma_{sn}^{\ell} - \sigma_{sn}^{u} = \frac{dT}{ds}$$

where σ_{nn}^{β} is the stress component acting normally
to the local surface in fluid β ($\beta=u,\ell$);

σ_{sn}^{β} is the stress component acting tangentially
to the surface in fluid β;

T is the local value of the surface
tension; and

$\frac{d\theta}{ds}$ is the local radius of curvature.

These equations are perfectly general. Their meaning can
best be appreciated with a simple example. Consider a
small barrier moving steadily across the surface of the
water, as depicted in Figure 2.2. Assume that the
barrier just touches the surface, and that immiscible

Barrier Moving to Right
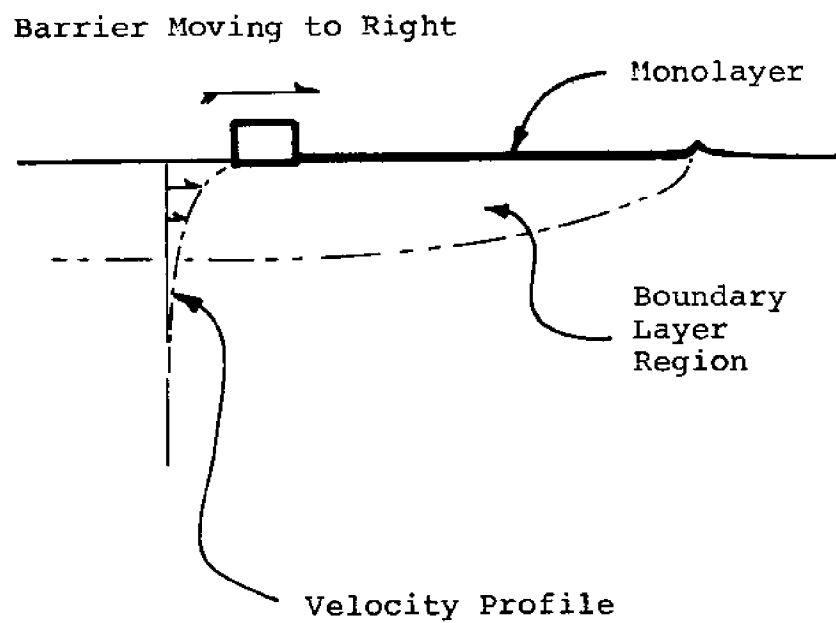
FIGURE 2.2

CAPTURED MONOLAYER PROBLEM

surfactants have collected in front of the barrier. Neglect variations in the 'z' direction. If the problem is in a steady state then the velocity everywhere within the surface film must be zero relative to the barrier. Relative to the barrier, the water will appear to be streaming from right to left, and if we neglect the normal stress boundary condition, the problem will look very much like a flat plate moving to the right on the surface of the water. In this case, the tangential stress is known to vary with x according to the following formula (see for example Schlichting (68) equation (7.32)):

$$\sigma_{sn}^{\ell} = \rho u_o^2 \ (.332) \sqrt{\frac{\nu}{u_o(L-x)}} \qquad (2.5)$$

Equating this with $\frac{dT}{ds}$ (which is now $\frac{dT}{dx}$) in equation 2.4, and integrating, we find the following formula for the variation in surface tension over the captured surface film;

$$T(x) = T_o - 2\rho u_o^2 (.332) \sqrt{\frac{\nu (L-x)}{u_o}} \qquad (2.6)$$

If the film is of the gaseous type (for example) then we can equate the local surface tension with the local

molecular area, whence:

$$A(x) = \frac{KT}{2\rho u_o^2 (.332)} \sqrt{\frac{u_o}{\nu(L-x)}} \qquad (2.7)$$

Since the maximum drop in surface tension is not infinite, but is, in fact, determined by the substance chosen for the surfactant, the formula will only hold for ranges of velocities and lengths. In fact, the drop in surface tension or alternatively, the total surface pressure, must equal exactly the total force, (i.e., the stress integrated over the area), applied by the water, thus:

$$\pi_{max} = 2\rho u_o^2 (.332) \sqrt{\frac{\nu L}{u_o}} \qquad (2.8)$$

where π_{max} is the maximum surface pressure observed for a tightly packed layer of the surfactant. Figure 2.3 shows the variation with x of T and A, for this problem.

If the total stress applied to the film exceeds π_{max}, then the film will contract, falling back on the barrier. Immediately in front of the barrier a zone of bulk phase surfactant will form, and, assuming the surfactant to be in the liquid state, internal gravity induced flows will develop so as to balance the stress
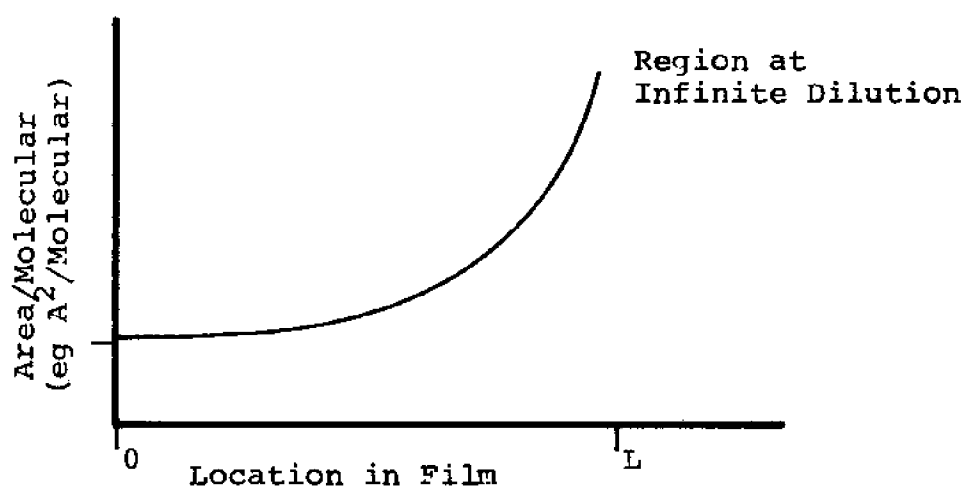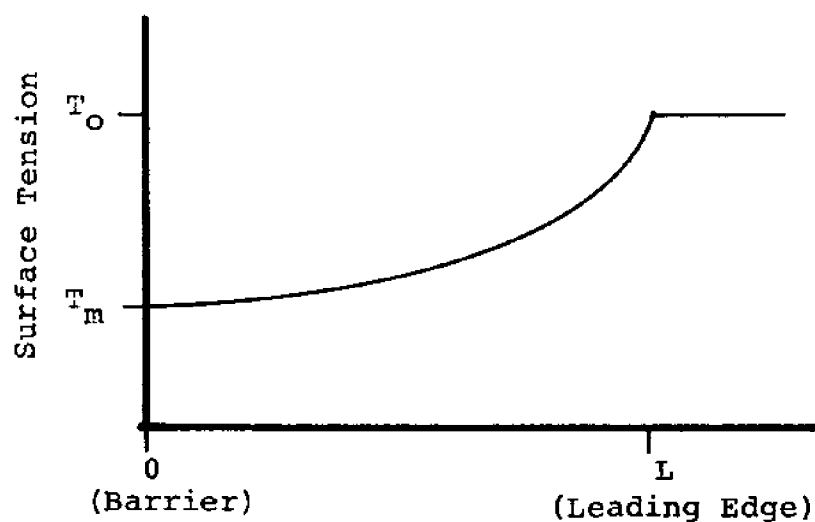
FIGURE 2.3

SKETCH OF SURFACE TENSION AND MOLECULAR AREA AS
FUNCTIONS OF LOCATION IN CAPTURED FILM

applied by the water below.

If we were to consider the normal stress boundary condition then things get more complicated. Preliminary laboratory observations suggest the formation of a sharp cusp at the leading edge stagnation point. The cusp appears to have a very small radius of curvature, suggesting $T \frac{d\theta}{ds}$ is sufficient to balance the stagnation pressure in this region.

While I am unfamiliar with any laboratory studies that would verify the analysis and calculations made for the example discussed above, Vines (59) performed a somewhat analogous calculation and experiment. In his study, wind was blown across a trough of water covered with a monolayer. The wind boundary layer was well established, exerting a nearly constant stress every where on the water's surface. Vines found that the film pressure measured at the rear (leeward) edge of the film varied linearly with the distance between the film balance (an apparatus designed to measure film pressure) and the forward (windward) edge of the monolayer. The upwind edge was identified by talc powder sprinkled on the water's surface which was convected by the water's motion to the stagnation point at the beginning of the film.

There the talc accumulated in a visible line. Since

the integrated stress (the total force) varied linearly

with the length of the film (on account of the constancy

of applied stress with respect to location), this

observation is in accord with our discussion of the problem

and the calculations that led to equation (2.8).

While this example and the proceeding one are

rather simple, they serve as useful points of departure

for a general discussion of the complexities of the wave

problem. A key assumption in the calculation leading

to equation (2.8) is that a steady state was achieved.

In an actual problem, this may only be approximately

true, and then only for highly insoluble surfactants.

To the extent that the surfactants are soluble, a small

fraction will constantly be washed from the monolayer

by the water streaming past.   If the water does not

supply an equal amount of surfactants by convection

from an upstream source, then eventually the material

in the film will be totally extracted. If the water is

recirculated, then at some point a balance will be

reached, but the distribution of surfactant between the

bulk phase, the captured film, and adsorption on other

surfaces will be a complicated function of the geometry of

the apparatus and the physical properties of the surfactant.

An oil spill moving with respect to the ocean may never reach a steady state due to the extraction of the surfactants orginally in the spill. To the extent that the oil's surfactants are replaced by naturally occurring surfactants, a steady state may be obtained, but it is likely that the amount of naturally occurring surfactant is rather small (that surfactants are found under natural conditions at sea is not to be disputed, see Barger and Garrett (70)). Of course, the idea of 'steady' is a relative one and so if the time scales we are concerned with are wave periods on the order of seconds or less, then the film need be of reasonably constant properties only over periods of minutes for the system to appear 'steady' to the wave oscillations.

Some general idea of the time scale of this extractive process may be obtained from Davies and Rideal wherein they reported (see page 181) fractional loss rates of 5.2×10^{-4} sec^{-1} for $C_9F_{19}COO^-$ and 1.6×10^{-5} sec^{-1} for $C_{18}H_{31}NC(CH_3)_3^+$ from the air-water interface. These correspond to characteristic extraction times on the order of hours to days respectively. Thus even for some of the

more slowly occurring hydrodynamic processes, a steady
state approximation may be useful.

However, in addition to these slow transient
phenomena, there are for surfactants soluble in one or
both of the adjoining phases, fast occurring exchanges
of molecules between the adjoining bulk phases and
the interface. These exchanges can effect the dynamical
properties of a film. This is a phenomena that has long
been recognized, but little experimental data are
available to allow us to come to grips with the problem.
The branch of surface chemistry that concerns itself
with determining some of the key parameters for this
process falls under the general heading of distribution
studies. It is the goal of such studies to determine
where the surfactant resides in a multiple fluid system.
In an oil-water system, such a study would presumably
determine both the equilibrium concentrations of surfactants
in the oil and water respectively, and the surface
concentration of molecules at the interface. Unfortunately,
there is remarkably little information available on the
subject. So little, in fact, that most of the recent
papers on the subject begin with words to the effect that
adsorption at liquid/liquid interface has received little

attention (see for example, Aveyard et al (72), Aveyard et al (70) and Chatterjee et al (68)).

The importance of this subject can best be appreciated by application of the results of one of these papers to a hypothetical problem. Aveyard, Briscoe and Chapman (72) recently examined the adsorption of n-dodecanol on the interface between n-octane and water. Through direct measurements and theoretical extrapolations, they determined:

1. The interfacial tension as a function of the mole fraction of n-dodecanol (dissolved) in the n-octane;

2. The surface areas of the n-dodecanol molecules adsorbed on the interface as a function of n-dodecanol concentration in the n-octane; and

3. The surface pressure as a function of the area per molecule.

They did not examine the concentration of n-dodecanol in the water as this was known through previous studies. Figure 2.4 summarizes their findings.

Consider now a surface between n-octane and water which is alternately compressed and stretched. Assume that the horizontal length scales are large. As the surface

FIGURE 2.4

-30-

SUMMARY OF AVEYARDS' RESULTS FOR ADSORPTION OF n-DODECANOL

INTERFACIAL TENSION

$(\frac{DYNES}{CM})$

MOLE FRACTION OF n-DODECANOL IN n-OCTANE
$(x \ 10^4)$

AREA PER MOLECULE

$(\overset{\circ}{A}^2)$

$\beta \sim 10^{-1}$ $\beta \sim 10^{-4}$ $\beta \sim 10^{-1}$ $\sim 10^{-5}$ Γ

SURFACE CONCENTRATION

$MOLES/(CM)^2$
$(x \ 10^{11})$

A

CONCENTRATION OF n-DODECANOL IN n-OCTANE PHASE
$(MOLES/CC) \ x \ 10^6$

SURFACE PRESSURE

$(\frac{DYNES}{CM})$

AREA PER MOLECULE $(\overset{\circ}{A}^2)$
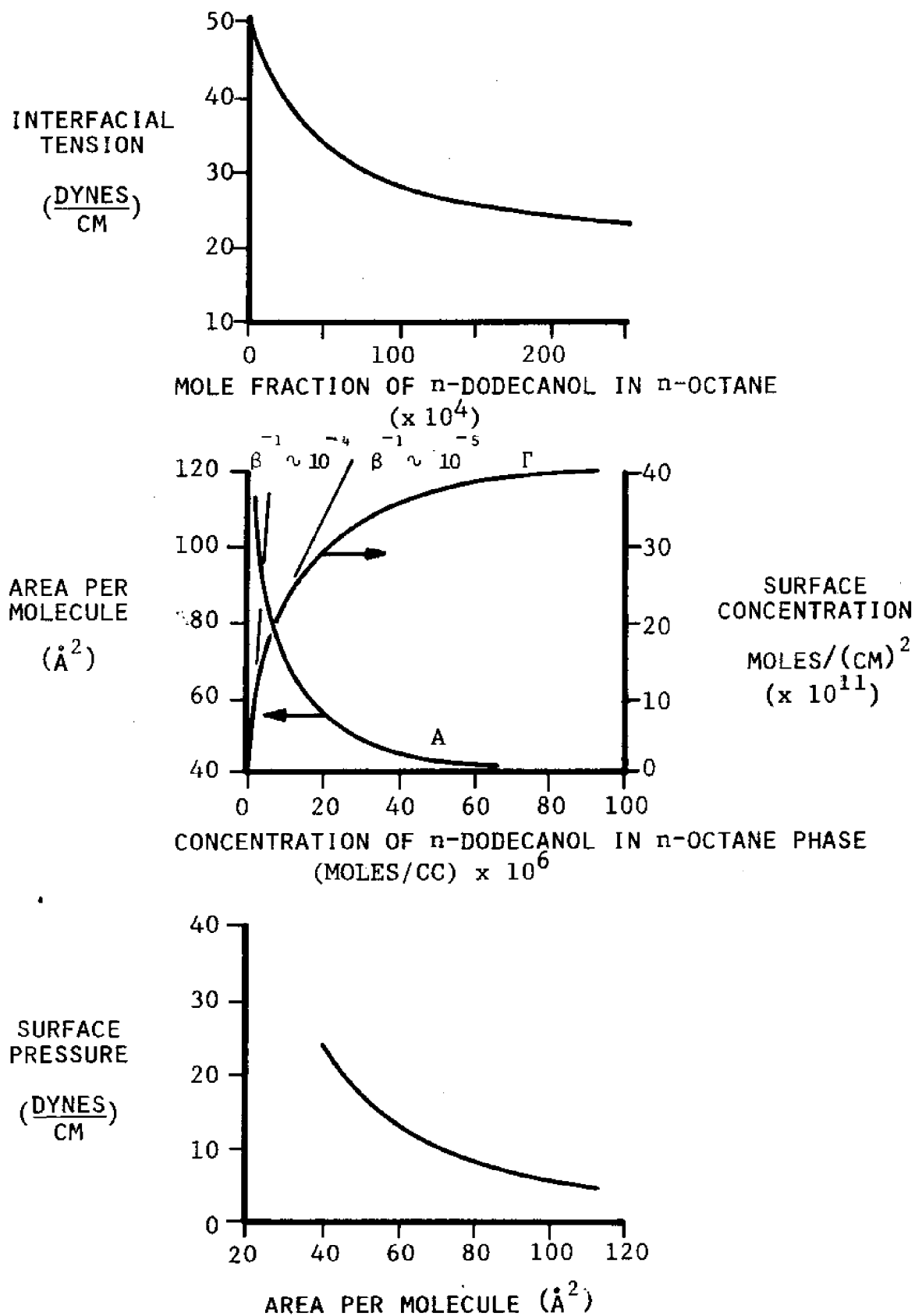
is compressed, molecules will pop out of the film,
creating a local bulk phase concentration that is in
equilibrium with the newly compressed state of the surface
(see Figure 2.4b). (We can neglect for now the molecules
that enter the water phase as they are a vanishingly
small fraction of the net exchange in the water-dodecanol-
octane system). As a result of this exchange of molecules
between the surface and the bulk phase, we will create
a concentration gradient of surfactant that will support
a net flux of material away from the interface under the
action of molecular diffusion. If the surface is
undergoing a steady sinusoidal oscillation, so that the area
per molecule is given by

$$A(+) = A_o + Re\{Ae^{i\sigma t}\}, \qquad (2.9)$$

then right at the interface, the bulk concentration would
also be varying with time. If the area oscillations were
small enough, then the concentration and area would be
approximately linear in each other, and

$$C(o,t) = C_o + Re\{C_{ie}^{i\sigma t}\} \qquad (2.10)$$

The diffusion equation in these circumstances becomes:

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial y^2} \qquad (2.11)$$

where y is the distance measured normally to the interface. The solution to equation 2.11 subject to the boundary condition (2.10) is the well known Stoke's formula:

$$C(y,t) = C_o + \text{Re}\{C_1 e^{i\sigma t - \frac{(1-i)y}{\sqrt{\frac{2_o D}{\sigma}}}}\} \qquad (2.12)$$

An important feature of the result above is that we can establish the instantaneous flux of molecules towards and away from the interface by evaluating $D \frac{\partial C}{\partial y}$ at $y = 0$.

If we now consider the equation for the conservation of surface active material adsorbed on the interface we have the general equation (see Levich 69.7)

$$\frac{\partial \Gamma}{\partial t} + \nabla \cdot (\Gamma \underline{V}_t - D_s \nabla \Gamma) + j_n = 0 \qquad (2.13)$$

where Γ is the surface concentration in
moles/cm^2 [i.e., $\Gamma = 1/NA$, $N =$
Avogadros number (6×10^{23})];

$\underline{v}_t$ is the tangential liquid velocity
at the surface;

D_s is a surface diffusion parameter
(we shall ignore this term); and

j_n is the flux of molecules to and
from the surface expressed in
terms of moles/cm^{2}sec.

In the problem at hand this equation becomes:

$$\frac{\partial \Gamma}{\partial t} + \frac{\partial}{\partial x}(\Gamma u) = -D \left.\frac{\partial c}{\partial y}\right|_{y=0} \qquad (2.14)$$

The oscillatory straining motion of the surface can be
modeled by selecting

$$u = Re\{u_1 e^{ikx-i\sigma t}\} \qquad (2.15)$$

where k is selected to be very small so that variations in
the 'x' direction can still be ignored in the diffusion
problem (allowing us to use Equation (2.12) with minor

modifications to calculate $\partial c/\partial y)$. Thus:

$$\frac{\partial \Gamma}{\partial t} + Re\{u_1 e^{ikx-i\sigma t}\}\frac{\partial \Gamma}{\partial x} + \Gamma Re\{ik\, u_1 e^{ikx-i\sigma t}\}$$

$$= -D \left.\frac{\partial c}{\partial y}\right|_{y=0} \tag{2.16}$$

Now let:

$$\Gamma = \Gamma_o + Re\{\Gamma_1 e^{ikx-i\sigma t}\} \tag{2.17}$$

and assume

$$c(x,0,t) = \beta\, \Gamma\, (x,t) \tag{2.18}$$

where β has units of cm^{-1}, and is given by the reciprocal of the slope of Figure 2.4b. Then Equation (2.16) becomes:

$$Re\{-i\sigma\Gamma_1 e^{ikx-i\sigma t}\} + Re\{u_1 e^{ikx-i\sigma t}\}Re\{ik\Gamma_1 e^{ikx-i\sigma t}\}$$

$$+ \Gamma_o Re\{iku_1 e^{ikx-i\sigma t}\} + Re\{\Gamma_1 e^{ikx-i\sigma t}\}Re\{iku_1 e^{ikx-i\sigma t}\}$$

$$= D \ Re\{- \frac{(1 - i)}{\delta} \ \beta \ \Gamma_1 e^{ikx-i\sigma t}\} \qquad (2.19)$$

where

$$\delta = \sqrt{2D/\sigma} \qquad (2.19a)$$

If we now assume that u_1 is small in some sense and that $\Gamma_1 << \Gamma_o$, then:

$$Re\{[-i\sigma\Gamma_1 + \Gamma_o iku_1 + (1 - i) \frac{\beta}{\delta} D\Gamma_1]e^{ikx-i\sigma t}\} \simeq 0 \qquad (2.20)$$

Thus,

$$\Gamma_1 = \Gamma_o \frac{k}{\sigma} u_1 \left[\frac{1}{1 - \frac{(1 - i)}{2} \beta\delta} \right] \qquad (2.21)$$

The factor $(\frac{k}{\sigma})^{-1}$ is the phase velocity of the compressional wave motion that we assumed for the horizontal velocity field. Thus, if we neglect the diffusion related term $(\frac{1 - i}{2})\beta\delta$, we can see from Equation (2.21) that the smallness condition imposed on u_1 is that u_1 (and thus the local particle velocities) be small compared to the wave phase velocity (i.e., $\frac{k}{\sigma} u_1 << 1$).

Equation (2.21) also tells us that the process of diffusion tends to reduce the magnitude of the oscillations of the surface concentration for a given oscillatory horizontal velocity field. This phenomena is known as relaxation. For "insoluble" surfactants, β is very small and this correction is negligible. However, in the example we are presently studying, β, the reciprocal of the slope of the Γ versus C curve in Figure 2.4b, is in the range of 10^4 to 10^5 cm^{-1}.

The parameter δ is determined by the diffusivity of the surfactant in the bulk phase and the frequency of the oscillation. The diffusivity of solute 1 in dilute solution in solvent 2 is given approximately by the equation:

$$D_{12} = 7.4 \times 10^{-8} \frac{(XM)^{1/2}T}{\mu v_1^{-6}} \qquad (2.22)$$

(see Reid and Sherwood (58), equation 8.34).

where   M = molecular weight of solvent

            (e.g., n-octane $\simeq$ 114 gm/mole)

          T = temperature

          μ = viscosity of solution (centipoise)

            ($\simeq$ 10 cp)

v_1 = molal volume of solute at normal

boiling point (cm^3/g mole)

X = "association" parameter of solvent,

which may be taken as approximately

1.0 for oils.

For dodecanal in n-octane at 300°K, this yields:

$$D_{12} \simeq 10^{-5} cm^2/sec \qquad (2.23)$$

Thus, we can establish a frequency at which the relaxation effect becomes of principal importance in the coupling of the interfacial straining motions and the surface concentration. Requiring, for example, that $\beta\delta < 10$, and choosing β to be 10^4, we have:

$$\beta\delta = 10$$

$$\sigma_R = \frac{2\beta^2}{10^2} D_{12} = 20 \qquad \text{rad/sec} \qquad (2.24)$$

More generally, we may consider M to be typically 3×10^2 gm/mole, T to be 3×10^2 °K, μ to be 10 to 10^2 cp, and v_1 to be 5×10^{-3} cm^3/gm-mole. Thus D_{12} is typically 10^{-6} cm^2/sec (or perhaps a factor of 10 smaller) and the

relaxation frequency, σ_R, is approximately:

$$\sigma_R \simeq 2 \times 10^{-8} \beta^2 \qquad (2.25)$$

Thus, if β is 10^4, then the relaxation effect is of prin-
cipal importance for frequencies below $\sigma_R \simeq 2$ rad/sec
(i.e., it attenuates the surface concentration oscillations
by about a factor of 10 or more relative to the straining
motions of the velocity field), but at higher frequencies
it becomes of lesser importance.

The presence of μ in the denominator of Equation
(2.22) suggests that the relaxation frequency diminishes
for more viscous oils.

The tangential stress applied at the interface
may be determined by relating the 'X' variations in the
surface concentration to the equivalent 'X' variations
in surface stress. That is:

$$\frac{\partial T}{\partial x} = \frac{dT}{d\Gamma} \frac{\partial \Gamma}{\partial x} \qquad (2.26)$$

But:

$$\frac{\partial \Gamma}{\partial x} = \text{Re}\{ik\Gamma_1 e^{ikx-i\sigma t}\} = \text{Re}\{\Gamma_0 (\frac{u_1}{u_0}) R\, ike^{ikx-i\sigma t}\} \qquad (2.27)$$

where $u_o = \frac{\sigma}{k}$, (the phase velocity of the

compressional wave) and

R = a complex relaxation factor;

R being defined by:

$$R = \frac{1}{1 - \frac{(1-i)}{2}\beta\delta} \tag{2.28}$$

If we assume that the variation of T with Γ is reasonably approximated by a straight line about the point Γ_o, then over the range of the oscillation

$$\frac{dT}{d\Gamma} = \text{constant} \equiv -\frac{E_o}{\Gamma_o} \tag{2.29}$$

where E_o is the surface elasticity (in the

same units as T, i.e., dynes /cm).

E_o is seen to be a positive, real

number.

Thus:

$$\frac{\partial T}{\partial x} = \text{Re}\{-E_o(\frac{u_1}{u_o})R\ ike^{ikx-i\sigma t}\} \tag{2.30}$$

But the variation in the local strain of the surface elements due to variation in the velocity field is given by:

$$e_x = \int_o^t \frac{\partial^2 u}{\partial x^2} \, dt = \text{Re}\{ik(\frac{ik}{-i\sigma})u_1 e^{ikx-i\sigma t}\}$$

$$= \text{Re}\{-ik(\frac{u_1}{u_o}) e^{ikx-i\sigma t}\} \tag{2.31}$$

(see Section 6)

Letting

$$ik(\frac{u_1}{u_o}) = e_s e^{i\theta x}, \quad \text{and} \tag{2.32}$$

$$E_o R = E_o |R| e^{i\theta R} = \frac{E_o}{(1 - \beta\delta + \frac{\beta\delta^2}{2})^{1/2}} e^{i\theta R} \tag{2.33}$$

then Equation (2.30) may be written as:

$$\frac{\partial T}{\partial x} = \text{Re}\left\{-\frac{E_o}{(1 - \beta\delta + \frac{\beta\delta^2}{2})^{1/2}} \frac{e^{i\theta R}}{} e_s e^{ikx-i\sigma t+i\theta x}\right\}$$

$$\frac{\partial T}{\partial x} = \frac{-E_o e_s}{(1 - \beta\delta + \frac{\beta\delta^2}{2})^{1/2}} \cos[kx-\sigma t+\theta_x+\theta_R] \tag{2.34}$$

where:

$$\theta_x = \tan^{-1}\left\{\frac{\text{Im}[ik(\frac{u_1}{u_o})]}{\text{Re}[ik(\frac{u_1}{u_o})]}\right\} \qquad (2.35)$$

$$\theta_R = \tan^{-1}\{\frac{-\beta\delta}{2-\beta\delta}\} \qquad (2.36)$$

If the surfactant is insoluble ($\beta\delta = 0$), then we can readily see that $\theta_R = 0$ and $|R| = 1$, and the tangential stress condition is given by E_o times the local derivative of the strain. As $\beta\delta$ grows, the coupling between the tangential stress and the straining of the surface diminishes, and a lag, θ_R, is introduced which causes the maximum stress at any position to be applied following the instant of maximum strain. At very large values of $\beta\delta$ (i.e, very low frequencies for any given system), the stress falls to $E_o/\beta\delta$ times e_s and the lag is 135°.

The problem in applying this model to the oil-water system is that we know virtually nothing about the β's or the E_o's for the complicated mixture of organic compounds that we call oils. This ignorance is best manifested by the complete absence of data that might simply suggest which surfactants are to be found in which oils.

Moreover, this simple treatment has not covered interactions between surfactant types. This could well be a critical feature of the oil problem because of the diversity of the chemical compounds found in oil. E.H. Lucassen-Reynders (73) (Part III) has shown that highly insoluble surfactants in the presence of more soluble surfactants cause interactions at the surface that have the effect of reducing the effective relaxation, thus increasing $E_o|R|$. Since there are bound to be surfactant species more insoluble than others in the oil mixture, this suggests that a simple knowledge of the available surfactant type is not sufficient. It appears that we require a thorough chemical analysis of several representative oils coupled with an examination of their dynamical properties if we are to come to grips in a firm, quantitative fashion with this surface elasticity phenomena as it might apply to the oil-water system.

However, our ignorance of the exact behavior of the oil-water system does not support the notion that there is no monolayer-induced surface elasticity in the oil-water system. There is, for example, a wealth of common knowledge relating to the calming effect of oil on waves that suggests such effects may be of great importance. In fact, we show in section 7 that in the absence of

surface elasticity, there can be no strong damping of waves by oil, and so the damping of waves by oil may be taken as a reasonable validation of the probable importance of surface elasticity. To accommodate this situation we shall therefore simply assume that the surface elasticity exists and examine a range of values for this parameter. Due to the complicated nature of the relaxation phenomena, we shall also assume that $\beta\delta$ is very small, i.e., that the surfactant is either a very slow diffuser or of very low solubility. As we shall see in the subsequent sections, the analysis is still rather complicated, and even if the requisite data existed, such assumptions would be warranted in any initial development.

Since oil is of a varied composition, we shall also examine three oil types which we call "light", "medium" and "heavy" and which are defined in Table 2.1. Some substantiation may be found for the hypothesized range of elasticities from Table 5-IV in Davies and Rideal (1961, page 265).

TABLE 2.1

GENERIC OIL TYPES

	Light	Medium	Heavy
Property (60°F)			
Viscosity (cp)	20	10	100
Density (gm/cc)	.8	.85	.9
Oil-air tension (°/cm)	30	30	30
Oil-water tension (°/cm)	20	20	20
Range of Elasticities (°/cm)	3-300	3-300	3-300

Most oils will find their counterpart in one of the three types. For example, Arabian Light crude oil is very much like the "Medium" oil.

This discussion of the interfacial elasticity and the possible complications that might arise in the oil spill problem has to this point been directed at the more general characteristics of monolayer covered interfaces. However, in addition to the difficulties associated with the diffusion related relaxation phenomena and the possible interactions of mixtures of surfactants, there are some particular complications that arise when we consider the application of these strain dependent surface tensions to the surface wave problem. In particular, the surface tension induced differences between the normal stresses applied by the upper and lower fluids on the interface may, in general, have a nonvanishing, steady second order term. This term arises from the possible synchronization of the time variations of the local radius of curvature with the oscillations of the local surface tension. In ordering our problem we would normally require that the oscillations of the surface tension be small compared to the mean value. This would have the effect of causing any steady components to be relatively small and thus negligible to our analysis.

However, for many surfactants the initial rate of change of surface tension would be very large. That is, the elasticity could be as large as the surface tension or larger. In this case, the extrapolation to the higher order term associated with the synchronization discussed above would suggest the presence of a non-negligible steady quantity. Actually, of course, the surface tension would only change a small amount before the straightline approximation implicit with the use of an elasticity would cease to be valid. That is, sinusoidal variations in strain would not lead to simple sinusoidal variations in the local surface tension, but would in fact lead to highly chopped wave forms for the surface tension, that would be rich in higher harmonics and perhaps have a net average offset. In short, the range of values hypothesized in Table 2.1 for E is not compatible with the interfacial tension selected for the oil-water interface if we anticipate carrying these results to second order in the parameter that measures the wave induced strain.

However, a lot of problems arise in carrying the analysis to second order, this being but one of them. Our approach will therefore be to look at the first order problem in happy ignorance of these complications. We

shall then restrict our discussion of the second order
problem to those very general features of the problem that
may well be insensitive to these complications. The only
alternative is to develop the surface tension versus local
strain in a Fourier series compatible with the actual
surface tension behavior, and this would introduce compli-
cations that would soon overwhelm us. This is, however,
a mathematically interesting phenomenon and subsequent
investigators may well find some useful analysis following
this alternative to its conclusion.

3. GENERAL PROPERTIES OF UNCONSTRAINED OIL SPILLS ON THE OCEAN

The interaction of an unconstrained oil spill with waves on the surface of the ocean is a subject that is best approached with a good understanding of the experimental observations presently available. To attempt otherwise, to theorize without the benefit of the observations, is to become mired at an early point in the multitude of competing and complementary physical processes at work within this very complicated physical system. In fact, in apparent contradiction of the theoretical complexity, actual oil spills exhibit some remarkably consistent features despite wide variations in oil properties and ambient conditions. We will subsequently develop our hypotheses regarding the oil spill's interaction with waves based upon some of the features so identified. The validity of a model so formulated will thus rest upon its ability to explain the features not hypothesized.

The best experimental observations of unconstrained oil spills in the open ocean for our purposes are those reported by Hollinger, et al (73a), (73b), (74). These spills were conducted by the U.S. Coast Guard in cooperation with the Virginia Institute

of Marine Sciences (VIMS), the NASA Wallops Island Station,

and the Naval Research Laboratory in Washington, D.C.

The petroleum derivatives tested included No. 2 fuel oil

(home heating oil), No. 4 fuel oil (a higher boiling

fraction suitable for many light industrial purposes,

erroneously identified by Hollinger as 'No. 4 crude oil'),

and No. 6 fuel oil (a heavy residual fuel oil used

typically by large power plants, again this oil was

mistakenly identified by Hollinger as 'No. 6 crude oil').

The spill volumes were in the range of 200 - 630 gallons,

and the tests were conducted in both moderate and

intermediate (white caps) sea states. The test sites

were in the Atlantic Ocean off the coast of Chesapeake

Bay. The oil spills were created either by draining

50 gallon drums floating in the water, or by discharging

the oil from a tank on the deck of a Coast Guard Cutter.

The release process typically took several minutes

suggesting that the initial spreading was not

characteristic of a point discharge, but rather of a

continuous release. In fact, because of the release

procedure, each spill tended to exhibit a rather unique

pattern that appeared to be related to the motion of the

ship during the discharge.

Observations of the spill's behavior were then

made from a Coast Guard helicopter and from a specially

equipped aircraft operated by the Naval Research Laboratory. In addition, samples were gathered from small boats to provide ground truth for the aerial observations.

The results so obtained included the area covered by the spill, the thickness profile of the spill, and the orientation of the oil spill with respect to the surface wind.

The thickness observations exhibited a characteristic feature for all tests irrespective of both the type of oil spilled and the initial pattern the oil spill acquired during the release process. Broadly speaking, it was found that the oil tended to separate into thin and thick regions. The thick region was on the order of 1 to 2 mm thick, $((10^{-1})\text{cm})$ while the thin region was typically several thousands of a millimeter thick $((10^{-4})\text{cm})$. The thick region contained 90%-95% of the total volume of oil spilled, while the thin region covered an ever increasing proportion of the area. The thickness in either of the two regions may have varied substantially. For example, Figure 44 in Hollinger (24) suggests variations of .4 mm to 1.7 mm for the thick region. Unfortunately these are averages over the beam spot which is on the order of 50 ft. in diameter. Thus such averages do not tell us much about

thickness variation with horizontal scales of less than
50 ft. On the positive side mass balance calculations
(see Hollinger (73a), page 44) based on the beam spot
averages could be counted on to determine the volume of
oil spilled to within 30% or so, and so we can be
reasonably confident that such thicknesses are represent-
ative of the broader scale thickness variations.

Figure 3.1 depicts four outlines of the inner
region of a spill of 630 gallons of No. 2 fuel oil at
times of 39 min., 76 min., 152 min., and 237 minutes
following the release of the oil. These outlines were
taken from Figure 29 of Hollinger (74). Notice that
while it is true that the inner region grows with time,
an entirely adequate first approximation would be that
this region remains of nearly constant dimension.

Hollinger also observed that the thicker
portions of the oil tended to be located on the leeward
edge of the slick. Since the oil spill was observed
to drift downwind relative to an observer fixed with
respect to water column, the thicker region thus lead
the thinner portion in drifting to leeward. Hollinger
attributes this behavior to the slight elevation of the
surface in the thick oil region, an explanation first
offered by Stroup (32) in 1932. Without going into the

TIME = 39 MINUTES AFTER RELEASE

AREA = 2300 m^2
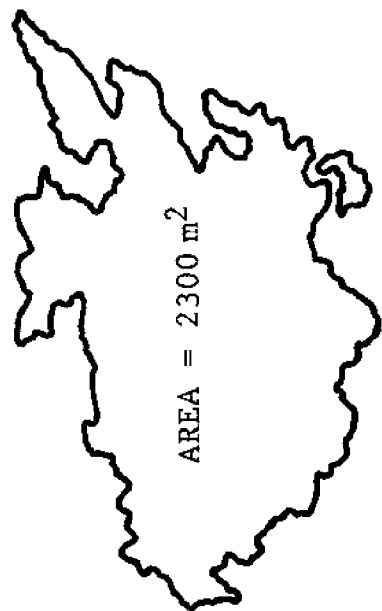
TIME = 76 MINUTES AFTER RELEASE

AREA = 2200 m^2

TIME = 152 MINUTES

AREA = 2800 m^2

TIME = 237 MINUTES

AREA = 3300 m^2

DISTANCE (m)

0   50   100
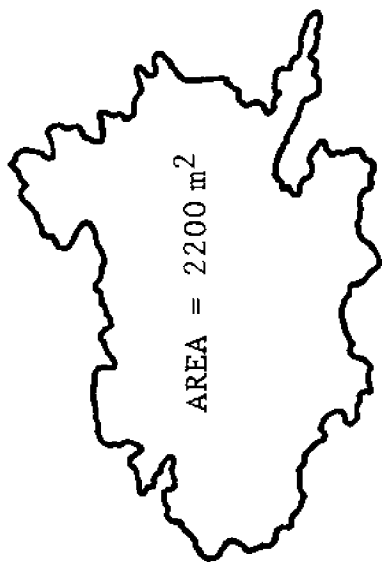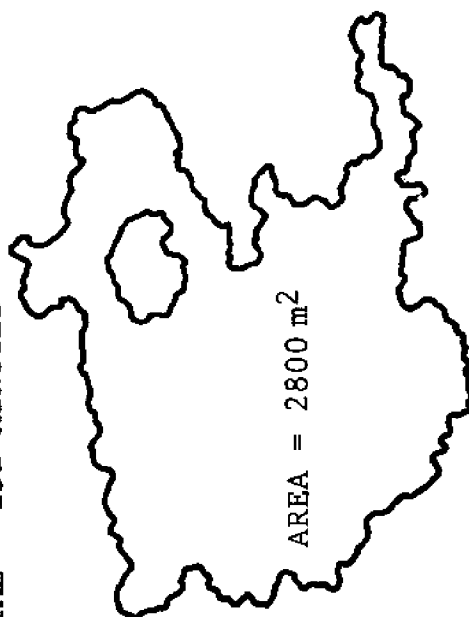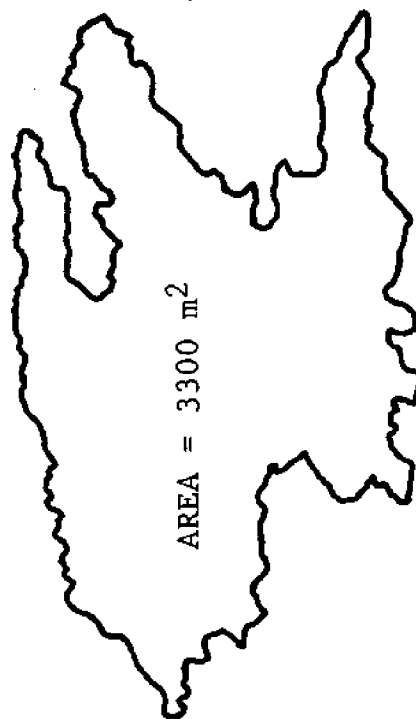
FIGURE 3.1   GROWTH OF THE INNER (THICK) REGION WITH TIME
(from HOLLINGER, 1974, Fig. 29, p. 62)

details, this explanation appears to be incorrect insofar
as the oil was behaving like a liquid. However, for
some of the very heavy oils, oils with pour points above
the ambient temperature, this explanation may be useful,
although such oils are typically so dense that they
would rise only 2% to 8% of their thickness above the
local sea surface elevation. This 'sail' idea might
also have application in the motion of small tar balls
on the ocean.

A complementary set of data was acquired for
a related spill test series by the Virginia Institute of
Marine Science (VIMS) under contract to the Environmental
Protection Agency, as reported in MacIntyre, Smith,
et al (74). Again measurements of the areal growth
of oil spills were made. In addition, the motion of the
spill relative to a 1 meter drogue was documented.

Figure 3.2 aggregates the VIMS data on the
area of #2 fuel oil spills with data drawn from Hollinger's
report for spills of #2 fuel oil in light winds (less
than 10 knots). Only Hollinger distinguished between
the thick and thin regions in the area measurements and
so the lower data points depicting the growth of the
thick region represents Hollinger's two experiments of
11 July 1972 and 17 May 1972. The upper grouping of data

FIGURE 3.2: AREA VERSUS TIME FOR NO. 2 FULE OIL SPILLS
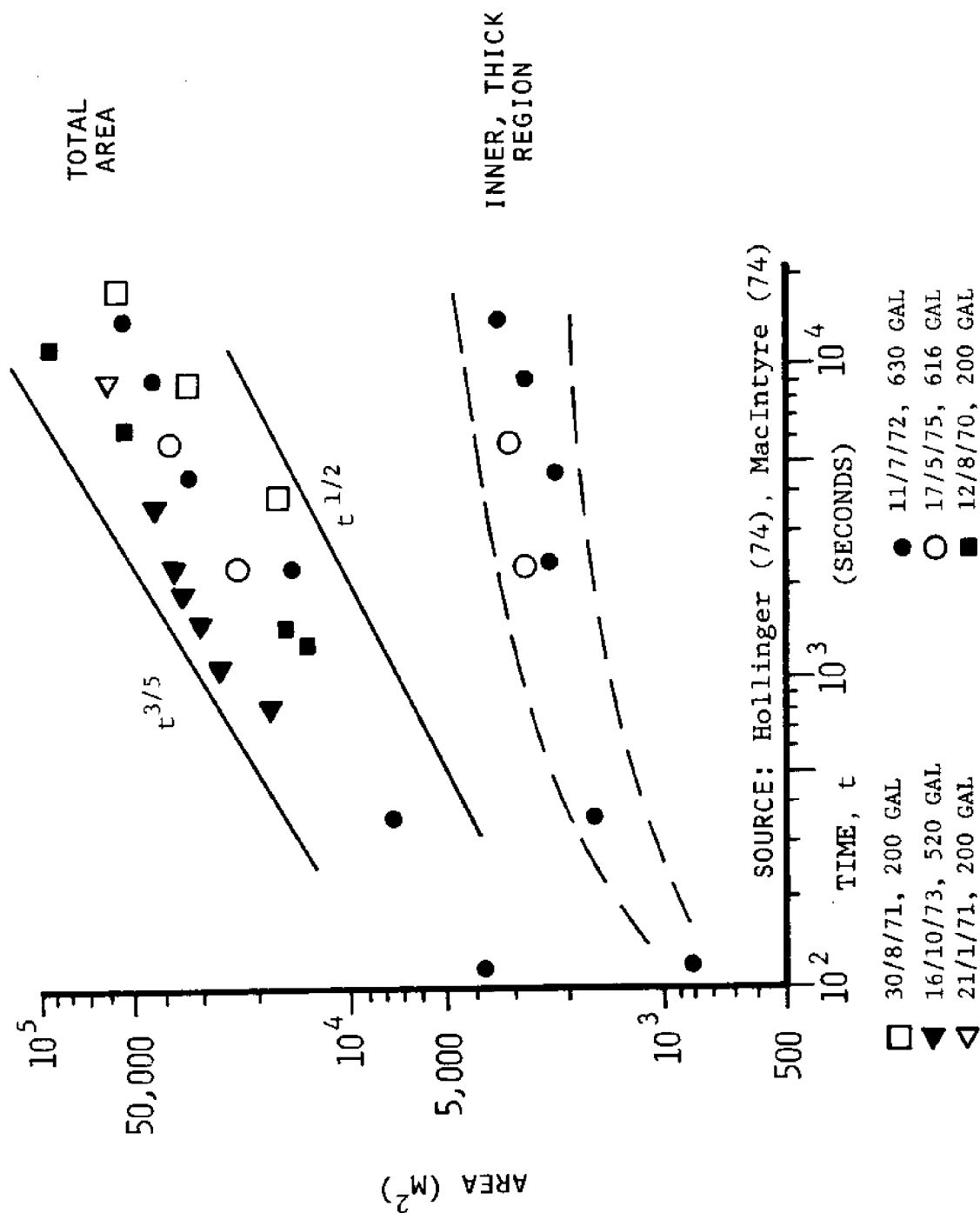
points includes both of Hollinger's experiments and the experiments reported by VIMS. The VIMS oil spills were of an initial volume of 200 gallons, while the NRL oil spills were of 616 gallons and 630 gallons respectively. Notice the characteristic slow rate of growth for the thick region. Also notice that despite the variations in spill conditions and oil spill volumes, the total area versus time behavior is approximately the same for all five spills.

Hollinger suggested a fit of $t^{3/5}$ through the upper data points (where t is time). As we can see, such a dependence is supported by the data although growth proportional to $t^{1/2}$ would seem to be just as acceptable. The VIMS investigators, working with but a subset of the data, came to the conclusion that a spreading law based on a $t^{3/2}$ power law was acceptable, but this seems most unlikely in view of the additional data presented here.

The VIMS investigators also established the fact that the oil tends to drift downwind more rapidly than a drogue constructed so as to track the motions of the top meter of water. They found that the divergence between the drogue's trajectory and that of the oil was relatively small for low wind speeds, increasing to 1% to 2% of the wind's velocity once the wind exceeds 10 or 12 knots.

Figure 3.3 is a photograph of a spill of number 4 fuel oil. This photograph was supplied through the courtesy of Mr. Richard Jadamec of the Coast Guard Research and Development Center in Groton, Connecticut. While it can't be seen from a single photo, the oil is overtaking dyed surface waters, although the dye release capsules make sufficient leeway to stay ahead of the spill. No attempt is made here to quantify this phenomena's dependence upon wind velocity, although the wind during this photograph was only about 10 knots, suggesting VIMS' observed threshhold of 10-12 knots for this leeway effect may be an artifice of their experimental method.

Visual observations made from small boats indicate that the oil in the thick region is contained in lenses of a few centimeters in diameter. Hollinger (74, page 54) reports that in simultaneous spills of No. 2 and No. 6 fuel oils, it was observed that the No. 6 fuel oil formed larger and thicker lenses relative to the No. 2 fuel oil. The reported thicknesses for the No. 2 fuel oil were at considerable variance to other experiments, however, so this should not be regarded as conclusive.
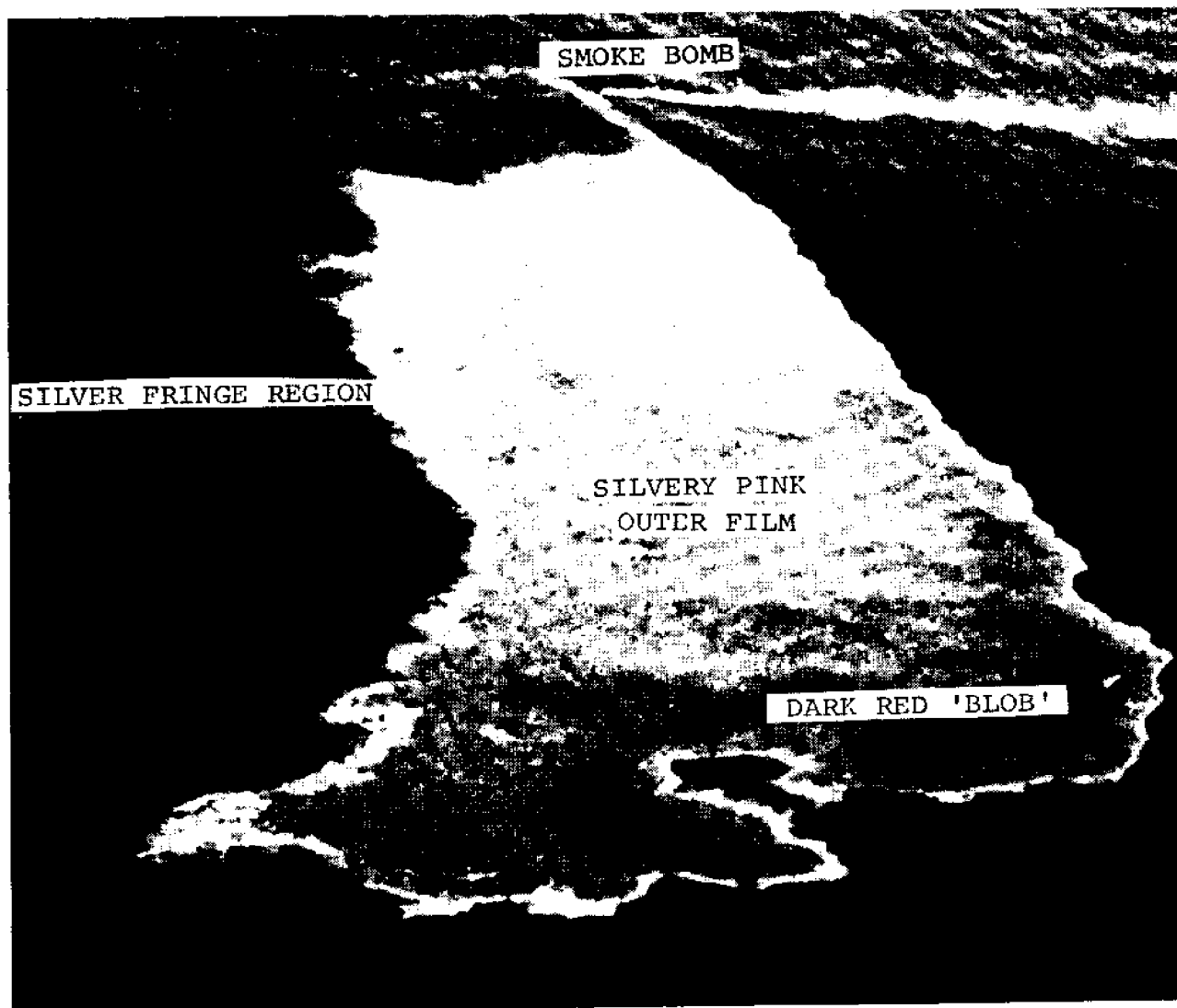
FIGURE 3.3

No. 4 fuel oil spill 30 minutes after release.
Wind is 10 knots. Black and white print obtained
from 35 mm color slide provided through the
courtesy of Mr. Richard Jadamec, U.S. Coast
Guard, R&D Center, Groton, Connecticut.

Hollinger also reported that in at least one experiment conducted in 10 knot winds with No. 4 fuel oil, the oil was initially very much dispersed during the release process, but following the withdrawal of the ship from the spill site, the dispersed thicker regions "appeared to pull together into a rather compact thick region surrounded by a much thinner slick". (See Figure 43 of Hollinger (74) and text of page 84).

This summary of observed oil spill behavior is reasonably exhaustive at least with respect to recent U.S. field experiments. The principal ommissions are some area versus time results for oils other than No. 2; area versus time behavior for No. 2 fuel oil obtained in higher wind speeds; and some earlier studies by Guinard (70) that suggested a qualitatively different spreading behavior, but which are of questionable validity due to the uncertainties in the interpretation of the remote sensing data.* There are also some data relating to accidental oil spills but in such cases the uncertainties regarding volume spilled, rate of release, etc. are quite large and interpretation is difficult.

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*Guinard's data was obtained using remote sensing techniques that were greatly improved upon in the subsequent studies by Hollinger.

With this disclaimer of universality, the following model appears to be consistent with the evidence at hand for an unconstrained oil spill in non-breaking waves once the spill is sufficiently old so as to be beyond the influences of the method of release.

1. The spill will be composed of two regions, one region being very thin with few inclusions of thick oil. The other region will be several orders of magnitude thicker on the average, and it may be composed of many small lenses of oil compacted so as to be nearly continuous.

2. The thick region will lead the thinner region in drifting downwind, and over time, it will exhibit nearly constant dimensions. The thin region will spread upon the ocean causing the area of the spill to exhibit a time dependence like $t^{1/2}$. Based upon the consistency of the data shown in Figure 3.2, which were obtained from spills varying in volume from 200 to 630 gallons, it appears plausible to expect the total area to

grow essentially independent of the
volume released, at lease for spills
100-1000 gallons in range.

3. The characteristic thickness for the
thick region is in the range of one or
two millimeters. Variations in
thickness of a factor of 2 or 3 might
be expected even if discrete lenses aren't
formed. The average thickness might be
dependent upon the characteristics of the
spilled oil, oils with properties like
those of the No. 6 fuel oil forming
thicker blobs than those like No. 2 fuel
oil.

4. If we combine both Hollinger's
observation that the thick region tends
to be reformed after an initial dispersion
and the observed low rate of growth of
this region with time, then we must conclude
that the formation of the thick region is
not simply a short lived transitional stage
in some highly unsteady spreading process.
Rather, we must suspect that some internal
dynamic is at work that tends to stabilize

this geometry, at least in the moderate
sea states investigated.

Thus, in considering wave interactions with an
oil spill it would seem sufficient to begin with oil
layers of constant dimensions, neglecting entirely the
problems associated with the spreading phenomena. If
such a model exhibits features that are compatible
with a stable thick region geometry and an enhanced
downwind drift velocity for the thick region, then we
might have learned something. We should also keep in
mind the possible temporal changes that will occur
in the physical properties of the oil as it "weathers".
This is to say that while it is clearly beyond the
scope of this study to look in detail at weathering
problems it can do no harm to be aware that as time
passes the more volatile compounds will be evaporating
into the air, just as the surface active compounds will
be washed from the oil-water interface into the bulk
of the water.

The following three sections serve as an
introduction to the matched assymptotic technique.
The initial problem discussed is the water-air monolayer
problem as it is possible to relate this problem to

previously published results. Section 7 first

specializes the technique to the oil spill problem.

4. EXPANSION, LINEARIZATION AND NON-DIMENSIONALIZATION OF THE GOVERNING EQUATIONS

Throughout this study we shall limit our discussion to two fluid systems. Thus the air-oil-water system is modeled as an oil-water system in which hypothesized stress conditions are applied at the upper oil surface. The ideas presented can, in principle, be generalized to more complex cases, but the algebra grows combinatorially, and the relative simplicity of the present analysis is quickly obscured.* We shall introduce the first order solution technique by considering the classical monolayer problem. This corresponds to the air water problem with a specialized tangential stress interfacial boundary condition. This simplifies some of our initial calculations. Also in the interest of keeping the analysis simple and consistent with the problem at hand, we assume that within any given region the fluids are homogeneous, incompressible and Newtonian. Further, we assume the depth of the underlying fluid (the water) is infinite.

---

*In general the restriction to two fluids is a limitation of little importance for water-oil-air systems due to the low density of the air compared with either the water or oil phases.

There are two non-dimensional lengths that will prove to be of great importance in this problem. The first is the non-dimensional wave amplitude, $\alpha = aK_o$, where a and K_o are as defined in Figure 4.1. The second is the non-dimensional Stoke's boundary layer depth, $\varepsilon = \delta K_o$. Here δ is the characteristic depth of penetration of an oscillating vorticity applied at a surface, diffusing into an irrotational core. On dimensional grounds, $\delta = (\nu/\sigma_o)^{.5}$ where ν is the kinematic viscosity and σ_o is the frequency of the oscillation. Thus δ varies from one region to the next depending on the kinematic vicosity of the adjacent fluids. It can also be seen that ε is the reciprocal of the square root of the wave Reynolds number since:

$$\varepsilon = (\nu K_o^2/\sigma_o)^{.5} = (\nu K_o/C_o)^{.5},$$

where $\quad C_o = \dfrac{\sigma_o}{K_o} = $ phase velocity.

Our discussion focuses upon the large Reynolds number $(\varepsilon^{-2} \gg 1)$ small amplitude case $(\alpha \ll \varepsilon \ll 1)$. In this problem it is assumed that the oscillations of the fluid interface(s) are so gentle that particle

(VERTICAL POSITION OF INTERFACE GREATLY
MAGNIFIED FOR EXPOSITIONAL REASONS)

$+\infty$ Y

CASE 1

Complex Wave Number

g

CASE 2

Complex Frequency

ρ^u, ν^u

λ

a

t

a   X

λ

ρ^ℓ, ν^ℓ

$K = \dfrac{2\pi}{\lambda}$

Direction of Wave
Propagation

$-\infty$
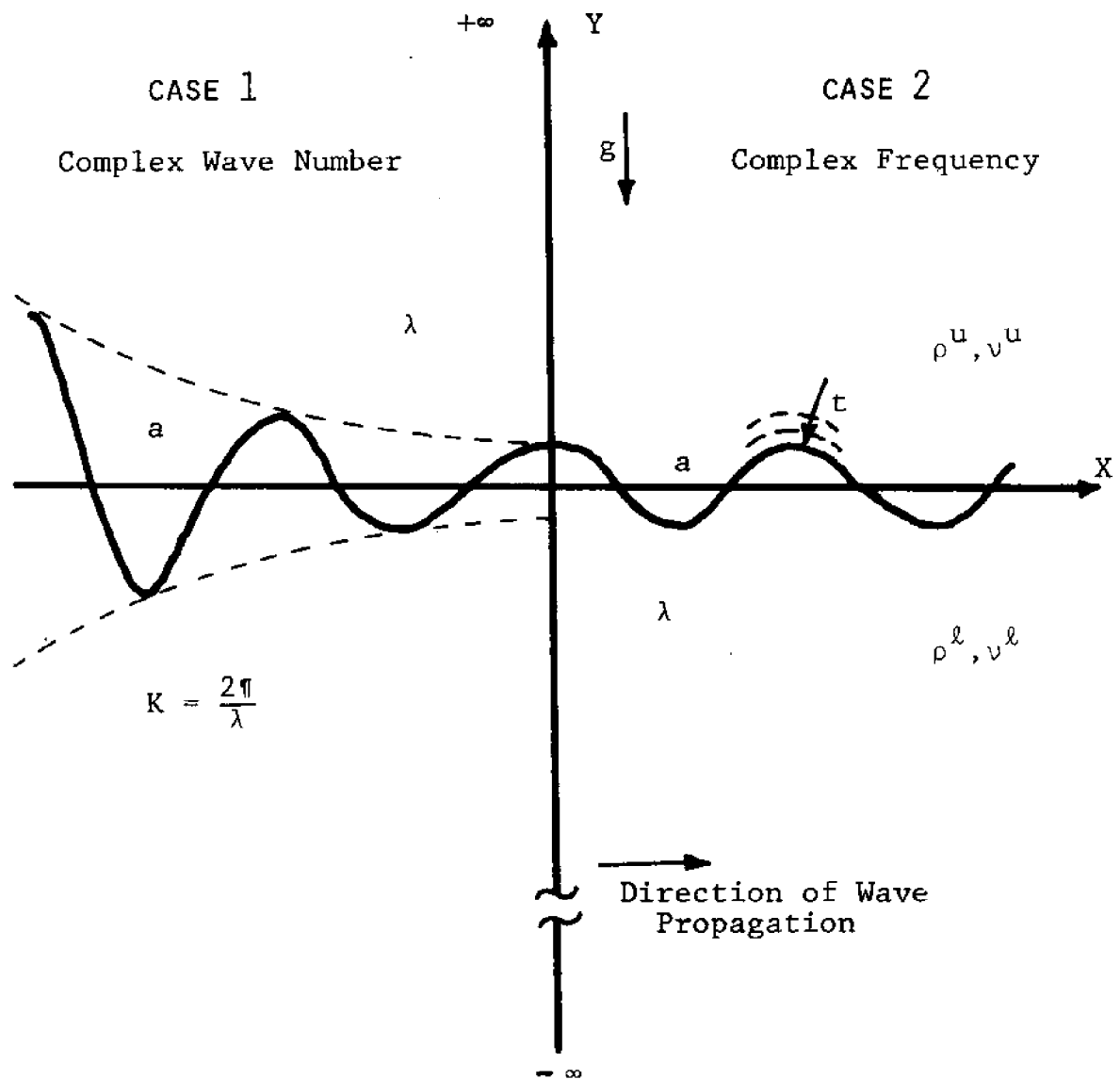
displacements on the interface are small compared to both the characteristic boundary layer depths and the wavelength. This allows us to evaluate boundary conditions known to apply at the interface in terms of a Taylor's expansion about the equilibrium level. This is the classical method of linearizing the boundary conditions.

In the first problem to be considered the two fluids shall be assumed to occupy the entire (X,Y) plane. The interface lying between the two fluids will be assumed to have both an interfacial tension and an elasticity consistent with our understanding of the behavior of monolayers at the air/water interface. This case will be called the monolayer problem. Since the discussion of the governing equations is nevertheless generalizable to the final oil film problem in which the upper fluid is of finite depth, these arguments will not be repeated.

The coordinates are chosen so that the horizontal direction is associated with the X axis, and the vertical direction will correspond to the positive Y axis. We shall limit our discussion to two-dimensional waves (no 'Z' dependence). The equilibrium level of the interface in this problem lies

about the plane Y=0. Two cases will be considered, one in which the time dependence is perfectly sinusoidal (the wavemaker/wave tank problem); the other in which the X dependence is perfectly sinusoidal (the classical decaying wave problem). In the first case, the irrotational motion will be found to have both sinusoidal and exponential 'X' dependencies (the wave number is complex). In the latter case, the frequency will be a complex number, leading to a time decaying sinusoidal motion. The coordinate system is sketched in Figure 4.2.

In the upper and lower boundary layer regions formed at the interface Y=0, the velocities will be given by the stream function Ψ^u and Ψ^ℓ respectively, where:

$$(u^\beta, w^\beta) = (\Psi^\beta_y, -\Psi^\beta_x);\qquad(4.1)$$

the β denoting u (upper) or ℓ (lower) respectively.

In the irrotational core regions, the velocities will be associated with the potential functions Φ^u and Φ^ℓ, where

$$(u^\beta, w^\beta) = (\Phi^\beta_x, \Phi^\beta_y).\qquad(4.2)$$

FIGURE 4.2

NON-DIMENSIONALIZED COORDINATES

REGION OF OVERLAP

$$\lim_{\epsilon^u \to 0} (y = 0^+_+, \quad y = \infty)$$

$$y = (\epsilon^u)^{-1} y$$

-2π

$+1$

$+2$

2π

$$y = -(\epsilon^\ell)^{-1} y$$

REGION OF OVERLAP

$$\lim_{\epsilon^\ell \to 0} (y = 0^=, \quad y = \infty)$$
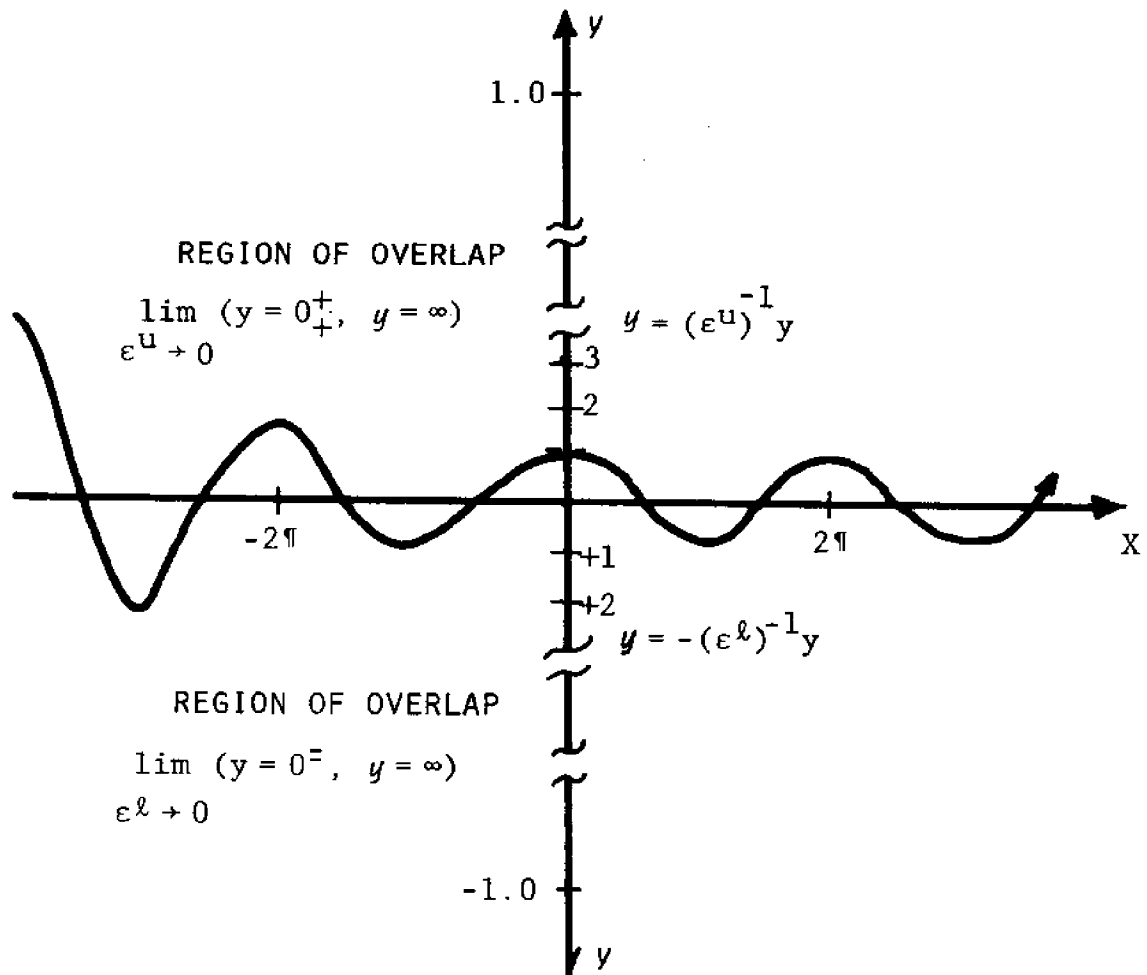
In the irrotational core regions, the particle trajectories will be approximately circular. Thus in the vicinity of the surface a particle will travel a distance of $2\pi a$ in the course of one wave oscillation. At the same time, a point of constant phase on the surface will travel a distance λ. Thus the ratio of the particle velocity to the phase velocity is equal to $\frac{2\pi a}{\lambda}$, which equals α. This shows α not only to be related to the non-dimensional wave amplitude, but also to be a measure of the size of the particle velocities relative to the wave phase velocity. It serves, in fact, as the primary ordering parameter for the inviscid flow properties.

The non-dimensionalization is performed in the outer core regions using the following time, length and speed scales:

$$\text{Time} = \sigma_o^{-1}$$
$$\text{Length} = K_o^{-1}$$
$$\text{Speed} = C_o$$

In the boundary layer regions, the scales are selected as follows:

$$\text{Time} \; = \; \sigma_o^{-1}$$

$$\text{Length} \begin{cases} \text{X direction:} \quad K_o^{-1} \\ \text{Y direction:} \quad \varepsilon^\beta K_o^{-1}, \quad \beta = u, \ell. \end{cases}$$

$$\text{Speed} \; = \; C_o$$

The rationale for the selection of this particular inner boundary layer, y-directed length scale is as follows. We assume that the interfacial motions will be the source of $O(\alpha)$ vorticity perturbations, having characteristic horizontal scales that correspond to the interfacial boundary oscillations $(O(K_o^{-1}))$. Under these circumstances, the terms in the vorticity equation will have the following relative orders of magnitude (factor out a term $O(\omega)$).

$$\frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \nu \left[\frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial x^2} \right]. \tag{4.3}$$

$$O(\sigma_o) + O\left(\alpha C_o \left\{ \frac{K_o}{K_o \varepsilon^{-1}} \right\} \right) = O\left(\frac{\nu}{\delta^2} \right) + O(\nu K_o^2)$$

where δ is the characteristic vertical dimension, and the velocity is $O(\alpha C_o)$.

Since α is assumed small for the waves in question, $(\alpha \ll \varepsilon \ll 1)$, the dominant terms are:

$$\sigma_o \sim \nu \delta^{-2} [1 + K_o^2 \delta^2]. \tag{4.4}$$

Imposing the constraint that $K_o^2 \delta^2 \ll 1$, we thus require:

$$\delta^2 \sim \frac{\nu}{\sigma_o} \tag{4.5}$$

or

$$\delta \sim \left[\frac{\nu}{\sigma_o}\right]^{1/2} = K_o^{-1} \left[\frac{\nu K_o}{C_o}\right]^{1/2} = K_o^{-1} \varepsilon \tag{4.6}$$

The requirement that $(K_o \delta)^2 \ll 1$ is thus seen to be equivalent to requiring $(\varepsilon)^2 \ll 1$, which is consistent with the assumed asymptotic formulation.

The non-dimensionalized governing equations for the stream functions in the upper and lower boundary layer regions are taken from the vorticity equation, which in two dimensional flow has the dimensional form:

$$\left[\frac{\partial \omega'}{\partial t} + u'\frac{\partial \omega'}{\partial x} + w'\frac{\partial \omega'}{\partial y}\right] = \nu^\beta \left[\frac{\partial^2 \omega'}{\partial x^2} + \frac{\partial^2 \omega'}{\partial y^2}\right] \tag{4.7}$$

Letting

$$u' = \alpha C_o \frac{1}{\varepsilon^\beta} \Psi_y \tag{4.8a,b,c}$$

$$w' = -\alpha C_o \Psi_x$$

$$\omega' = -K_o \alpha C_o \left[\frac{1}{\varepsilon^{\beta 2}} \Psi_{yy} + \Psi_{xx}\right].$$

where Ψ is $O(1)$, as are Ψ_x and Ψ_y. We readily find the following non-dimensional form (Note that in the lower boundary layer w' will now be defined as positive downwards

so as to preserve the form of the governing equation in both upper and lower regions):

$$\left[\frac{\partial}{\partial t} + \frac{1}{\varepsilon^\beta}\, \Psi_y\, \frac{\partial}{\partial x} - \frac{1}{\varepsilon^\beta}\, \Psi\, \frac{\partial}{\partial y}\right]\left[-\frac{\Psi_{yy}}{\varepsilon^{\beta 2}} - \Psi_{xx}\right] =$$

$$\varepsilon^{\beta 2}\left[-\frac{\Psi_{yyyy}}{\varepsilon^{\beta 4}} - 2\,\frac{\Psi_{yyxx}}{\varepsilon^{\beta 2}} - \Psi_{xxxx}\right]$$

(4.9)

where, $\mathscr{y} = \{\begin{smallmatrix} y/\varepsilon^u \\ y/\varepsilon^\ell \end{smallmatrix}\}$, y being the non-dimensional outer variable $(y = K_o Y)$. (See Figure 4.2). The script $\mathscr{y}$ will not carry the u,ℓ superscript as the correct identification will always be obvious from the accompanying notation. We can see from this equation that the asymptotic expansion for the stream function in the upper boundary layer must at least incorporate terms in ε^u, while in the lower boundary layer, the expansion must at least contain terms in ε^ℓ. The upper and lower boundary layer stream functions are related in part by the kinematic conditions requiring continuity of velocity at their common interface. In very general terms, this therefore suggests that the appropriate asymptotic expansions for the upper and lower stream functions must contain terms in both ε^u and ε^ℓ.*

---

*The idea here is to treat ε^u and ε^ℓ as independent variables. Actually, of course, for any two given superposed fluids, variations in ε^u will be reflected in variations in ε^ℓ since they both share a common dependence on the only adjustable parameters, C_0 and K_0. However, this fiction is convenient for the bookkeeping and so we shall retain it.

Likewise the kinematic conditions linking the outer irrotational velocities to the boundary layer velocities suggest that the outer potential functions must be expanded in a power series in both ε^u and ε^ℓ. Thus, suitably general asymptotic power series expansions for the stream functions corresponding to rightward traveling waves are of the form:

$$\psi^u = Re\{\alpha e^{iKx-i\sigma t}[\psi^u_{oo}(y) + \varepsilon^\ell\psi^u_{01}(y) +$$

$$\tag{4.10}$$

$$\varepsilon^u\psi^u_{10} + \varepsilon^{\ell 2}\psi_{02}(y) + \varepsilon^u\varepsilon^\ell\psi_{11}(y)...]\} + O(\alpha^2)$$

$$\psi^\ell = Re\{\alpha e^{iKx-i\sigma t}[\psi^\ell_{oo} + \varepsilon^u\psi^\ell_{01} + \varepsilon^\ell\psi_{10} +$$

$$\tag{4.11}$$

$$\varepsilon^{u 2}\psi_{02} + ...]\} + O(\alpha^2)$$

where:

	Case 1	Case 2	
K	$1+\varepsilon^\ell K_{10}+\varepsilon^u K_{01}+...$	1	(4.12a,b)
σ	1	$1+\varepsilon^\ell\sigma_{10}+\varepsilon^u\sigma_{01}+...$	(4.13a,b)

and the α dependence is explicitly included in the formula for $\psi^{u,\ell}$ at variance with the more general formulae of equations 4.1 and 4.2.

The potential functions must satisfy Laplaces equation:

$$\Phi_{yy} + \Phi_{xx} = 0 \qquad (4.14)$$

Corresponding to the assumed X, and t dependence they may thus be written:

$$\Phi^u = Re\{\alpha e^{iKx-i\sigma t}[A_{oo}^u + \epsilon^\ell A_{01}^u + \epsilon^u A_{10}^u + \dots]e^{-Ky}\} +$$

$$O(\alpha^2) \qquad (4.15)$$

$$\Phi^\ell = Re\{\alpha e^{iKx-i\sigma t}[A_{oo}^\ell + \epsilon^u A_{01}^\ell + \epsilon^\ell A_{10}^\ell + \dots]e^{+Ky}\} +$$

$$O(\alpha^2) \qquad (4.16)$$

where the A_{ij}^β are constants related to the kinematic dynamic boundary conditions.

Inserting the assumed forms for the stream functions in equation (2), we find the following linearized equation for the $O(\alpha)$ terms:

$$\frac{1}{(\epsilon^\beta)^2}\left[\psi^{\beta IV} + i\sigma\psi^{\beta II}\right] - (K)^2\left[2\psi^{\beta II} + i\sigma\psi^\beta\right] +$$

$$(\epsilon^\beta K^2)^2\psi^\beta = 0 \qquad (4.17)$$

This can be decomposed further by equating like powers of ε^u and ε^ℓ, with the results:

Case 1

$$\psi_{oo}^{\beta IV} + i\psi_{oo}^{\beta II} \equiv \mathcal{L}\{\psi_{oo}^{\beta}\} = 0. \tag{4.18}$$

$$\mathcal{L}\{\psi_{ij}^{\beta}\} = 0; \quad i=0,1; j=0,1,2\ldots \tag{4.19}$$

$$\mathcal{L}\{\psi_{20}^{\beta}\} = 2\psi_{oo}^{\beta II} + i\psi_{oo}^{\beta}. \tag{4.20}$$

etc.

Case 2

$$(\sigma = 1 + \varepsilon^u \sigma_{01} + \varepsilon^\ell \sigma_{10} + \ldots)$$

$$\mathcal{L}\{\psi_{oo}^{\beta}\} = 0. \tag{4.21}$$

$$\mathcal{L}\{\psi_{01}^{\beta}\} = \begin{cases} -i\sigma_{01}\psi_{oo}^{\beta II} \ , & \beta = \ell & (4.22a) \\ \\ -i\sigma_{10}\psi_{oo}^{\beta II} \ , & \beta = u & (4.22b) \end{cases}$$

$$\mathcal{L}\{\psi_{10}^{\beta}\} = \begin{cases} -i\sigma_{01}\psi_{oo}^{\beta II} \ , & \beta = u & (4.23a) \\ \\ -i\sigma_{10}\psi_{oo}^{\beta II} \ , & \beta = \ell & (4.23b) \end{cases}$$

etc.

where β is used to denote u (upper) or ℓ (lower) as desired.

The general solutions to the ordinary differential equations defined above are:

Case 1

$$\psi^{\beta}_{ij} = a^{\beta}_{ij} + b^{\beta}_{ij}\, y + c^{\beta}_{ij}\, e^{-\frac{(1-i)}{\sqrt{2}}y} + d^{\beta}_{ij}\, e^{\frac{(1-i)}{\sqrt{2}}y} ;$$

(4.24)

$$\beta = u, \ell; \quad i = 0, 1; \quad j = 0,1,2,\ldots$$

$$\psi^{\beta}_{20} = a^{\beta}_{20} + b^{\beta}_{20}\, y + c^{\beta}_{20}\, e^{-\frac{(1-i)}{\sqrt{2}}y} + d^{\beta}_{20}\, e^{-\frac{(1-i)}{\sqrt{2}}y} +$$

$$\frac{1}{2} a^{\beta}_{oo}\, y^2 + \frac{1}{6} b^{\beta}_{oo}\, y^3 - c^{\beta}_{oo}\, \frac{(1-i)}{2\sqrt{2}} y\, e^{-\frac{(1-i)}{\sqrt{2}}y} +$$

$$d^{\beta}_{oo}\, \frac{(1+i)}{2\sqrt{2}}\, y\, e^{\frac{(1-i)}{\sqrt{2}}y} .$$

(4.25)

etc.

Case 2

(4.26)

$$\psi^{\beta}_{oo} = a^{\beta}_{oo} + b^{\beta}_{oo}\, y + c^{\beta}_{oo}\, e^{-\frac{(1-i)}{\sqrt{2}}y} + d^{\beta}_{oo}\, e^{\frac{(1-i)}{\sqrt{2}}y} .$$

$$\psi_{01}^{u} = a_{01}^{u} + b_{01}^{u}\, y + C_{01}^{u}\, e^{-\frac{(1-i)}{\sqrt{2}}\, y} +$$

$$d_{01}^{u}\, e^{\frac{(1-i)}{\sqrt{2}}\, y} - C_{oo}^{u}\, \sigma_{10}\, \frac{(1+i)}{2\sqrt{2}}\, y\, e^{-\frac{(1-i)}{\sqrt{2}}\, y} +$$

$$d_{oo}^{u}\, \sigma_{10}\, \frac{(1+i)}{2\sqrt{2}}\, y\, e^{\frac{(1-i)}{\sqrt{2}}\, y} . \tag{4.27}$$

etc.

We have thus constructed the general solutions to the first order problem. We have now to specialize the coefficients so as to satisfy the pertinent boundary conditions.

5. BOUNDARY CONDITIONS AND MATCHING TECHNIQUES FOR THE MONOLAYER PROBLEM

The fluid motions associated with the rightward traveling interfacial waves are presumed to die out as we get farther and farther away from the interface. This behavior has been insured with the y dependence selected for the potential functions Φ^u, Φ^ℓ (equation 4.15 and 4.16).

Corresponding to our expansions in ε^u and ε^ℓ for the wavenumber, potential functions and stream functions, we shall define the interface with the function $F(x,t)$ as follows:

$$F(x,t) =$$

$$\mathrm{Re}\{\alpha e^{iKx-i\sigma t}[\xi_{oo} + \varepsilon^\ell \xi_{10} + \varepsilon^u \xi_{01} + \ldots]\} + O(\alpha^2) = y$$

(5.1)

At the interface, we require the normal stress to be discontinuous by an amount related to the product of the surface tension and the local curvature of the interface. We also require the tangential stress to be discontinuous by an amount related to the (tangential) variations in the surface tension as induced by gradients of the surface concentration of surface active molecules. These statements may be formulated in dimensional units as follows:

$$\left\{ \sigma_{nn}^{\ell} - \sigma_{nn}^{u} \right\}_{y=F} = T \frac{1}{R} \qquad (5.2)$$

$$\left\{ \sigma_{sn}^{\ell} - \sigma_{sn}^{u} \right\}_{y=F} = E\xi_s \qquad (5.3)$$

where:

σ_{nn}^{i} is the component of fluid stress at the boundary of region i acting normally to the interface $(dynes/cm^2)$;

σ_{sn}^{i} is the component of fluid stress at the boundary of region i acting tangential to the interface $(dynes/cm^2)$;

T   is the surface tension (dynes/cm);

R   is the radius of curvature (cm) of the interface;

E   is the modulus of elasticity (dynes/cm); and

ξ_s is the gradient of the normalized surface concentration taken along the interface. It has dimension (cm^{-1}), and is entirely analogous

> to longitudinal variations in
> the strain of a beam in the commonly
> encountered longitudinal wave
> problem.

The normal and tangential stresses can be evaluated in terms of the x and y derivatives of the horizontal and vertical velocities and the local pressure as follow (see Weyhausen and Latone (60)):

$$\sigma_{nn}^i = - p^i \Big|_{y=F} + 2\mu^i \frac{1-F_x^2}{1+F_x^2} \frac{\partial w^i}{\partial y} \Big|_{y=F} +$$

$$2\mu^i \frac{F_x}{(1+F_x^2)} [u_y + w_x] \tag{5.4}$$

$$\sigma_{sn}^i = \mu^i \frac{(1-F_x^2)}{(1+F_x^2)} \left\{ \frac{\partial u^i}{\partial y} + \frac{\partial w^i}{\partial x} \right\}_{y=F} -$$

$$\frac{F_x}{(1+F_x^2)} 4\mu^i \frac{\partial u^i}{\partial x} \Big|_{y=F} \tag{5.5}$$

Inserting these forms into equations (10) and (11), non-dimensionalizing using the inner boundary layer scales coupled with the stress $\rho_\ell C_o^2$, and introducing the following non-dimensional coefficients, we have

$$\left\{-p + qp^u\right\}_{y=F} + \omega_g F(1-q) -$$

$$2\left[\frac{1-F_x^2}{1+F_x^2}\right]\left\{\epsilon^\ell \frac{\partial w^\ell}{\partial y} + q\epsilon^u \frac{\partial w^u}{\partial y}\right\}_{y=F} +$$

$$2\frac{F_x^2}{(1+F_x^2)}\left\{\epsilon^{\ell 2}\left[\frac{\partial w^\ell}{\partial x} - \frac{1}{\epsilon^\ell}\frac{\partial u^\ell}{\partial y}\right] - q\epsilon^{u 2}\left[\frac{\partial w^u}{\partial x} + \frac{1}{\epsilon^u}\frac{\partial u^u}{\partial y}\right]\right\}_{y=F} = \omega_\sigma \frac{F_{xx}}{(1+F_x^2)}3/2 \quad (5.6)$$

$$\frac{(1-F_x^2)}{(1+F_x^2)} * \left[(\epsilon^\ell)^2\left\{\frac{\partial w^\ell}{\partial x} - \frac{1}{\epsilon^\ell}\frac{\partial u^\ell}{\partial y}\right\}_{y=F} - \right.$$

$$\left. q(\epsilon^u)^2\left\{\frac{1}{\partial^u}\frac{\partial u^u}{\partial y} + \frac{\partial w^u}{\partial x}\right\}_{y=F}\right] -$$

$$\frac{F_x}{1-F_x^2} 4 \frac{\partial u^u}{\partial x}\left[\epsilon^{\ell 2} - q\epsilon^{u 2}\right] = \zeta\xi_s \quad (5.7)$$

where

$$q \quad = \quad \rho^u / \rho^\ell \quad , \qquad \text{the ratio of densities;}$$

$$\zeta \quad = \quad EK/\rho_1 C_o^2 \quad , \qquad \text{the Weber number;}$$

$$\omega_g \quad = \quad g/KC_o^2 \quad , \qquad \text{a Froude number (eg., gravity waves correspond to } \omega_g = 1);$$

$$\omega_\sigma \quad = \quad TK/\rho^\ell C_o^2 \quad , \qquad \text{(eg., capillary waves correspond to } \omega_\sigma = 1);$$

$$p^\beta \quad = \quad \frac{(p^\beta + \rho^\beta gy)}{\rho_\beta C_o^2} \quad , \qquad \text{the non-dimensional dynamic pressure.}$$

and the velocities u^β, w^β are positive in the sense of the outer (x,y) variables.

The dynamic boundary conditions, (5.6) and (5.7), are linearized by expanding about the equilibrium level, $y = 0$. Expanding and rewriting in terms of the inner

variables, y, the resulting order α equations are determined to be:

$$\left\{-p^{\ell} + q^{pu}\right\}_{y=0} + \omega F_g(1-g) -$$

$$2\left\{\epsilon^{\ell}\frac{\partial w^{\ell}}{\partial y} + q\epsilon^{u}\frac{\partial w^{u}}{\partial y}\right\}_{y=0} = \omega_\sigma F_{xx} \qquad (5.8)$$

$$(\epsilon^{\ell})^2\left\{\frac{\partial w^{\ell}}{\partial x} - \frac{1}{\epsilon^{\ell}}\frac{\partial u^{\ell}}{\partial y}\right\}_{y=0} -$$

$$q(\epsilon^{u})^2\left\{\frac{\partial u^{u}}{\epsilon^{u}\partial y} + \frac{\partial w^{u}}{\partial x}\right\}_{y=0} = \zeta\xi_s \qquad (5.9)$$

where we have taken account of the sign change in the y variable for the lower fluid, (u^{β}, w^{β} being positive wrt (x,y)).

The dynamic pressures, the p^{ℓ} and p^{u} terms in equation (5.8), are known in either of the irrotational core regions from Bernoulli's equation, which in our non-dimensional variables has the form:

$$- \hat{p}^{\beta} = \phi_t^{\beta} + \frac{1}{2}\left[\phi_x^{\beta^2} + \phi_y^{\beta^2}\right]. \qquad (5.10)$$

where the hat, ^, denotes the irrotational value. The Navier Stokes equation may be utilized to determine the variation of the dynamic pressure within the boundary layers. Combining these observations we may evaluate $p^{\ell}\big|_{y=0}$ and $p^{u}\big|_{y=0}$ with the equations:

$$p^{\ell}\bigg|_{y=0} = \hat{p}^{u}\bigg|_{y=-\varepsilon^{\ell}y_{\infty}} + \int_{y_{\infty}}^{0} \frac{\partial p}{\partial y}\, d \qquad (5.11)$$

$$p^{u}\bigg|_{y=0} = \hat{p}^{u}\bigg|_{y=+e^{u}y_{\infty}} + \int_{y_{\infty}}^{0} \frac{\partial p}{\partial y}\, d \qquad (5.12)$$

where y_{∞} denotes the horizontal plane in which the inner and outer solutions are to be joined.

The value of the irrotational component of the dynamic pressure at the matching point $y = \varepsilon^{\beta}y_{\infty}$ may be determined by expanding $\hat{p}^{\beta}$ about y=0 in a Taylor series. Inserting these expansions into (5.11) and (5.12) we have the result that:

$$p^{\ell}\bigg|_{y=0} = \hat{p}^{\ell}\bigg|_{y=0} - \frac{\partial \hat{p}^{\ell}}{\partial y}\bigg|_{y=0} (\varepsilon^{\ell}y_{\infty}) +$$

$$\frac{1}{2} \frac{\partial^{2}\hat{p}^{\ell}}{\partial y^{2}}\bigg|_{y=0} (\varepsilon^{\ell}y_{\infty})^{2} + \ldots - \int_{0}^{y_{\infty}} \frac{\partial p}{\partial y}\, dy. \qquad (5.13)$$

$$p^u\bigg|_{y=0} = \hat{p}^u\bigg|_{y=0} + \frac{\partial p^u}{\partial y}\bigg|_{y=0} (\varepsilon^\ell y_\infty) + \ldots -$$

$$\int_0^{y_\infty} \frac{\partial p}{\partial y} \, dy. \tag{5.14}$$

where, again, account has been taken of the sign of y in the lower boundary layer (5.13).

The order α equations determining the vertical pressure gradient within the boundary layers are:

$$\frac{\partial p^\ell}{\partial y} = \varepsilon^\ell \left\{ -\frac{\partial w^\ell}{\partial t} + (\varepsilon^\ell)^2 \frac{\partial^2 w^\ell}{\partial x^2} + \frac{\partial^2 w^\ell}{\partial y^2} \right\} + O(\alpha^2) \tag{5.15}$$

$$\frac{\partial p^u}{\partial y} = \varepsilon^u \left\{ -\frac{\partial w^u}{\partial t} + (\varepsilon^u)^2 \frac{\partial^2 w^u}{\partial x^2} + \frac{\partial^2 w^u}{\partial y^2} \right\} + O(\alpha^2) \tag{5.16}$$

Where w^ℓ is now considered positive downwards.
Thus, the integrands in the (5.13) and (5.14) may be evaluated as assumptotic series based on the inner stream function expansions (i.e., equation 4.10 and 4.11 coupled with equation 4.25 through 4.27).

The kinematic matching conditions require that the horizontal and vertical velocity vary continuously through the matching region. These conditions may be stated as follows:

$$\Phi^\beta_x \Bigg|_{y=\pm\epsilon^\beta y_\infty} = \frac{1}{\epsilon^\beta} \psi^\beta_y \Bigg|_{y=y_\infty}$$

<div align="right">(horizontal velocity)

(5.17)</div>

$$\Phi^\beta_y \Bigg|_{y=\pm\epsilon^\beta y\infty} = \pm\, \psi^\beta_x \Bigg|_{y=y_\infty} \quad ; \quad \beta = \{^\ell_u\}$$

<div align="right">(vertical velocity)

(5.18)</div>

Again the potential functions Φ^β_y and Φ^β_x, may be expanded about y=0 in a Taylor series. The stream functions Ψ^β_y, Ψ^β_x, are evaluated under the assumption that terms of the form $e^{(-(1-i)/\sqrt{2})}\, y\infty$ may be neglected. These techniques are consistent with the idea of the assumptotic nature of the power series expansions for the inner and outer solutions. For example if we choose $y_\infty = 1/\epsilon^{\frac{1}{2}}$, then for the inner variables we have $\ell im(y_\infty) = \ell im \epsilon^{-\frac{1}{2}} = \infty$ thus $e^{-y\infty/\sqrt{2}} \to 0$; while for the outer variable, $\ell im\ (\epsilon y_\infty) = \ell im\ (\epsilon^{\frac{1}{2}}) = 0$. Thus the matching point for the outer solution goes to zero as $\epsilon \to 0$.

These conditions provide a link between the a^β_{ij}, b^β_{ij} parameters of the inner solution and the A^β_{ij} parameters of the outer solution. They also imposed the requirement that $d^\beta_{ij} = 0$, i.e. no exponential growth in the boundary layers. Since there are many

equalities generated from these conditions, we shall not present them here with the exception of the important result that all $b_{oi}^\beta = 0$. (see Appendix A for a complete listing of these conditions). This springs from the horizontal velocity matching conditions, (equation 5.17) which produces terms order α/ε^β in the b_{oi}^β coefficients which can have no counterpart in the assumed outer solution.

The linearized order α kinematic boundary conditions following from continuity of velocity at interface are of the form:

$$\frac{1}{\varepsilon^u}\ \psi_y^u\bigg|_{y=0} = \frac{1}{\varepsilon^\ell}\ \psi_y^\ell\bigg|_{y=0}$$

(horizontal velocity)

(5.19)

$$-\ \psi_x^u\bigg|_{y=0} = \psi_x^\ell\bigg|_{y=0}$$

(vertical velocity)

(5.20)

Notice that the sign on the right hand side of equation (5.20) is positive, consistent with the downward directed inner variable utilized in the lower boundary layer.

Again numerous equations result and they may be found in Appendix A. One immediately important finding is that:

$$b^\beta_{oi} - \frac{(1-i)}{\sqrt{2}} \, c^\beta_{oi} = 0 \qquad\qquad (5.21)$$

which when combined with the result discussed above $(b^\beta_{oi} = 0)$ provides us with the requirement that

$$\psi^\beta_{oi} = a^\beta_{oi} \qquad\qquad (5.22)$$

That is, the only terms that can exist in the Ψ^β expansion that are of order $(\alpha\varepsilon^{\beta 0})$ are constants, resulting in vertical velocities that are constant in y, although variable in X, at this order of ε^β

The procedure now is to evaluate 5.13 and 5.14 using the expansions of the stream and potential functions and equations 5.15 and 5.16. With the resulting expansion for the (non-dimensional) dynamic pressure, we can evaluate equation 5.8 in terms of an asymptotic power series in α, ε^u and ε^ℓ. This then provides us with a number of equations that can be readily simplified using the kinematic and matching conditions springing from (5.17) through (5.20). These steps will be sketched in the next section.

It is interesting at this point to compare this rather tedious, yet straight forward accounting

procedure with the classical eigenvalue technique.
Dorrestein (52a), solving the single fluid monolayer
problem (no overlying fluid) is faced at this point with
a fifth order equation (equation (22)) in frequency to
the half power (or fifth order in K) that is anything
but simple. Harrison (08), considering the two fluid
problem with no monolayer, is also faced with a fifth
order equation in $\sigma^{\frac{1}{2}}$ (fifth order in K, see page 398).
In theory both the above analyses are more general than
the present study since no restriction is made on the
wave Reynold's number. However, in order to make head-way
against these formidable algebraic problems, both authors
make a large Reynold's number assumption. Harrison lets ν
and ν' be small, while Dorrestein defines the parameter θ,
(which is identically the wave Reynold's number) and
notes $\theta << 1$ for waves of practical importance, allowing
him to continue by using small parameter expansions
(eg. equation (27)). Thus, in a practical sense we have
lost nothing by assuming from the beginning that
$\varepsilon^2 << 1$, and we have gained a great deal, both in the
simplicity of the calculations, and in the almost
clinical way in which the various properties can be
isolated and defined (eg. the dynamic pressure with
equations (5.13) and (5.15)).

6. FIRST ORDER PROPERTIES FOR THE MONOLAYER PROBLEM

Performing the indicated expansions and equating like powers of ε^u, ε^ℓ, we find the following hierarchy of equations for the order 'α' normal stress condition (5.8):

Case 1:   ($\sigma = 1$)

$$\varepsilon^o: \quad iA_{00}^\ell + iqA_{00}^u + \omega_g (1-q) \xi_{00}$$

$$= -\omega_\sigma \xi_{00} \tag{6.1a}$$

$$\varepsilon^\ell: \quad -iA_{10}^\ell + (iA_{00}^\ell + a_{00}^\ell) y_\infty + iqA_{01}^u$$

$$+ \omega_g (1-q) \xi_{10} = -\omega_\sigma (\xi_{10} + \xi_{00} 2K_{10}) \tag{6.1b}$$

$$\varepsilon^u: \quad -iA_{01}^\ell + q(iA_{10}^u - (a_{00}^u + iA_{00}^u) y_\infty)$$

$$+ \omega_g (1-q) \xi_{01} = -\omega_\sigma (\xi_{01} + \xi_{01} 2K_{01}) \tag{6.1c}$$

Case 2:   ($K = 1$)

$$\varepsilon^o: \quad iA_{00}^\ell + iqA_{00}^u + \omega_g (1-q) \xi_{00}$$

$$= -\omega_\sigma \xi_{00} \tag{6.2a}$$

$$\varepsilon^\ell: \quad -iA_{10}^\ell + i\sigma_{10}A_{00}^\ell + q(iA_{01}^u + i\sigma_{10}A_{00}^u)$$

$$+ \omega_g(1-q)\xi_{10} = -\omega_\sigma \xi_{10} \tag{6.2b}$$

$$\varepsilon^u: \quad -iA_{01}^\ell - i\sigma_{01}A_{00}^\ell + q(iA_{10}^u + i\sigma_{01}A_{01}^u)$$

$$+ \omega_g(1-q)\xi_{01} = -\omega_\sigma \xi_{01} \tag{6.2c}$$

If we now incorporate the kinematic matching and boundary conditions for the vertical velocity (see Appendix A) we determine:

$$\text{Case 1:} \quad (\sigma = 1)$$

$$1 = \frac{\omega_\sigma + \omega_g(1-q)}{1+q} \tag{6.3a}$$

$$K_{10} = \frac{C_{10}^\ell}{\xi_{00}(1 + q + 2\omega_\sigma)} \tag{6.3b}$$

$$K_{01} = \frac{-qC_{10}^u}{\xi_{00}(1 + q + 2\omega_\sigma)} \tag{6.3c}$$

Case 2:   (K = 1)

$$1 = \frac{\omega_\sigma + \omega_g(1 - q)}{1 + q} \tag{6.4a}$$

$$\sigma_{10} = -\frac{c_{10}^\ell}{2\xi_{00}(1 + q)} \tag{6.4b}$$

$$\sigma_{01} = \frac{qc_{01}^u}{2\xi_{00}(1 + q)} \tag{6.4c}$$

Accounting for the difference in nomenclature, Equation (6.4a) corresponds to Lamb's result in §267, Equation (2), for wave motions on the interface of two superposed fluids. The horizontal velocity boundary condition at the interface may be invoked to establish the relationship:

$$c_{10}^\ell = \sqrt{2}\,(1 + i)\xi_{00} + c_{10}^u \tag{6.4d}$$

where use is made of the result $b_{10}^\ell = -b_{10}^u = \xi_{00}$. We can now use the tangential stress condition to solve for either c_{10}^ℓ or c_{10}^u in terms of ξ_{00}.

Since we are limiting our discussion in this section to immiscible monolayers, we know that variations in the local surface concentration will be associated solely with straining motions in the surface. If u_m is

the component of the Eulerian velocity lying in the direction of the local tangent to the surface (denoted here by the unit vector $\underline{m}$), then:

$$u_m = \underline{u} \cdot \underline{m} \qquad (6.5)$$

If we now take the gradient of this scalar quantity, evaluated at the surface, in the direction of the surface tangent, we will determine the local Eulerian rate of strain of the surface whence:

$$\dot{e} = \frac{\partial u_m}{\partial S} = \{\underline{m} \cdot \nabla(\underline{u} \cdot \underline{m})\}_{y=F}$$

or,

$$\dot{e} = \{u_x \frac{(1 - F_x^2)}{(1 + F_x^2)} + \frac{F_x}{(1 + F_x^2)}(u_y + w_x)$$

$$+ \frac{F_{xx}}{(1 + F_x^2)^2} F_t\}_{y=F} \qquad (6.6)$$

Here, $\dot{e}$ denotes the rate of strain at the surface; $\underline{m} \cdot \nabla$ is the gradient of a scalar in the direction $\underline{m}$;

$$\underline{m} = (\frac{1}{(1 + F_x^2)^{1/2}}, \frac{F_x}{(1 + F_x^2)^{1/2}}) \qquad (6.7)$$

and use is made of the surface kinematic condition:

$$\{F_t = w - uF_x\}_{y=F} \tag{6.8}$$

Thus the strain of the surface at any time 't' will be given by the integral:

$$e = \int_0^t \frac{\partial u_m}{\partial S} \, dt + e_1(x) \tag{6.9}$$

where $e_1(x)$ is the initial state of strain of the surface. We can readily argue that $e_1(x)$ is of little importance in the wavemaker case (Case 1) because we can leave the wavemaker on long enough so that any initial variations in strain have relaxed via self induced oscillations. However, in the time decaying wave problem, it is possible to envisage pathologic states of initial strain that lead to decaying standing waves that make the problem dependent upon $e_1(x)$. These problems, however, have not been considered by previous investigators and since the purpose of this section is to demonstrate the solution technique by comparison with these earlier results, we will not take the matter any further. Thus, setting $e_1(x)$ to zero, we may determine that the variation in strain along the surface is given by:

$$\xi_s = \underline{m} \cdot \nabla e = \frac{1}{(1 + F_x^2)^{1/2}} e_x$$

$$+ \frac{F_x}{(1 + F_x^2)^{1/2}} e_y \qquad (6.10)$$

which to first order in α is simply:

$$\xi_s = \int_0^t \frac{\partial^2 u}{\partial x^2} dt = \frac{i}{\sigma} \frac{\partial^2 u}{\partial x^2} \qquad (6.11)$$

where we have invoked the time dependence of the $O(\alpha)$ velocities.

Thus, the highest order, non-zero components to the tangential stress condition provide us with the equation:

$$\epsilon^\ell c_{10}^\ell + \epsilon^u q c_{10}^u = \zeta [\xi_{00} + \frac{(1 - i)}{\sqrt{2}} c_{10}^u] \qquad (6.12)$$

But we also have the kinematic requirements of equation (6.4d) relating c_{10}^ℓ and c_{10}^u to ξ_{00}, so we can evaluate c_{10}^u and c_{10}^ℓ in terms of ξ_{00}.

Setting $\zeta = 0$ (no monolayer) we find:

$$c_{10}^u = -\xi_{00} \frac{\sqrt{2}(1 + i)}{1 + qQ} \; ; \; \text{and} \qquad (6.13)$$

$$c_{10}^{\ell} = \xi_{00} \frac{\sqrt{2}(1+i)}{1+qQ} qQ \qquad (6.14)$$

where:

$$Q \equiv \frac{\varepsilon^u}{\varepsilon^\ell} = (\frac{\nu^u}{\nu^\ell})^{1/2}$$

Inserting these values into equations (6.4b) and (6.4c) [Case 2], we find the following equation for the order ε damping:

$$\varepsilon^\ell \sigma_{10} + \varepsilon^u \sigma_{01} = \varepsilon^\ell (\sigma_{10} + Q\sigma_{01})$$

$$= -\varepsilon^\ell \frac{\sqrt{2}(1+i)}{(1+qQ)} \frac{qQ}{(1+q)} \qquad (6.15)$$

Rewriting this equation in dimensional units, we have:

$$\sigma_0 (\varepsilon^\ell \sigma_{10} + \varepsilon^u \sigma_{01})$$

$$= \frac{\sqrt{2}(1+i)K_0\sqrt{\sigma_0} \ \rho_\ell \rho_u (\nu_u \nu_\ell)^{1/2}}{(\rho_\ell + \rho_u)(\rho_u \nu_u^{1/2} + \rho_\ell \nu_\ell^{1/2})} \qquad (6.16)$$

in agreement with both Harrison (1908, page 399) and Johns (1968, equation 6.25).

Alternatively, if we retain the ζ parameter, we find:

$$c_{10}^{u} = \xi_{00} \frac{[m - \frac{\sqrt{2}(1+i)}{1+qQ}]}{[1 - \frac{(1-i)}{\sqrt{2}} m]} \qquad (6.17a)$$

$$c_{10}^{\ell} = \xi_{00} \frac{[\frac{(1-i)}{\sqrt{2}} \frac{qQ}{(1+qQ)} - m]}{[1 - m\frac{(1-i)}{\sqrt{2}}]} \qquad (6.17b)$$

where

$$m = \frac{\zeta}{\varepsilon^{\ell} + q\varepsilon^{u}} \qquad (6.18)$$

This parameter 'm' may be thought of as the square of the ratio of the longitudinal (compressive) wave phase speed to the transverse (gravity plus surface tension) wave phase speed. The mass entrained by the compressive motion is represented here by the epsilon terms in the denominator, (i.e., $\varepsilon^{\ell} + q\varepsilon^{u}$).

Summing the first order (ε^{1}) contributions to the frequency perturbation, we have:

$$\varepsilon^\ell \sigma_1 \equiv \varepsilon^\ell \sigma_{10} + \varepsilon^u \sigma_{01}$$

$$= \varepsilon^\ell \left\{ \frac{m(1 + qQ) - \frac{3+i}{\sqrt{2}} \frac{qQ}{1+qQ}}{2(1+q)(1 - \frac{(1-i)}{\sqrt{2}}m)} \right\} \qquad (6.19)$$

which reduces to the following form upon setting $q = 0$, [this is air-water problem as $q = \rho_{air}/\rho_{water} = O(10^{-3})$]:

$$\sigma_1 = \left\{ \frac{m(1 - m\frac{(1+i)}{\sqrt{2}})}{(1 - \sqrt{2}m + m^2)} \right\} \qquad (6.20)$$

The problem associated with this specialization of q, i.e., $q = 0$, corresponds to the problem studied by Dorrestein and if we make the following identifications,

$$\theta^{1/2} = \varepsilon^\ell \quad \text{and} \quad X = \zeta$$

so if:

$$\alpha_d \equiv \frac{X}{2\theta^{1/2}} \qquad \text{then} \qquad \alpha_d = \frac{m}{\sqrt{2}}$$

and

$$\varepsilon_{2im} \equiv \frac{1}{4}(2\theta)^{1/2} = \frac{\sqrt{2}}{4} \varepsilon^\ell \quad , \text{ then}$$

we can show that the imaginary part of Equation (6.20) is
the negative of:

$$\frac{\varepsilon_{2im}2\alpha_d^2}{(1 - 2\alpha_d + 2\alpha_d^2)} = Im\{Eq. \ 6.20\} \qquad (6.21)$$

in agreement with Dorrestein (see page 270 of [52a]).

The kinematic properties of the flow in the
vicinity of the interface are of some interest due to the
limiting form of c_{10}^{ℓ} as m goes to infinity. Evaluating
the horizontal velocity in terms of the lower boundary
layer expansion we have:

$$u^{\ell}(x,y,t) = Re\{\alpha e^{ikx - i\sigma t}[\psi_{10}' + \varepsilon^{\ell}\psi_{20}' + \dots]\}$$

$$+ O(\alpha^2)$$

$$= Re\{\alpha e^{ikx - i\sigma t}[b_{10}^{\ell} - \frac{(1 - i)}{\sqrt{2}} c_{10}^{\ell}$$

$$\cdot e^{[-(1 - i)/\sqrt{2}]y} + O(\varepsilon^{\ell})]\}$$

$$+ O(\alpha^2) \qquad (6.22)$$

From the matching condition applied to the vertical
velocity at $y = -\varepsilon^{\ell} y_{\infty}$, we can readily show that

$$b_{10}^{\ell} = iA_{00}^{\ell} = \xi_{00} \qquad \text{(See Appendix A)} \qquad (6.23)$$

Considering the air-water case (q→0) we can specialize
Equation (6.17b) for the case m → ∞, with the result:

$$\lim_{\substack{m \to \infty \\ q \to 0}} \{c_{10}^{\ell}\} = \xi_{00} \frac{(1 + i)}{\sqrt{2}} \qquad (6.24)$$

Thus the order α^1 horizontal velocity at the interface
approaches zero, since

$$\lim_{\substack{m \to \infty \\ q \to 0}} \{b_{10}^{\ell} - \frac{(1 - i)}{\sqrt{2}} c_{10}^{\ell}\}$$

$$= \xi_{00} (1 - \frac{(1 - i)}{\sqrt{2}} \frac{(1 + i)}{\sqrt{2}}) = 0 \qquad (6.25)$$

From a mathematical standpoint, there would seem
to be nothing too remarkable about this result, and in
fact if we invoke the asymptotic nature of our expansion,
then we should expect m to be very large, because

$$\lim_{\varepsilon^{\ell} \to 0} \{m\} = \lim_{\varepsilon^{\ell} \to 0} \{\frac{\zeta}{\varepsilon^{\ell}}\} \to \infty$$

However, for real fluids in conditions characteristic of
the problem we are considering, the parameter ζ is typ-
ically of the same order as ε^{ℓ}, or perhaps even smaller,
over the range of frequencies and wavelengths for which
the analysis has any validity. Thus, m is usually order
1 or smaller, and the appropriate specification of the
asymptotic nature of the expansion must include a
statement to the effect

$$\zeta = O(\varepsilon^{\ell} + q\varepsilon^{u})$$

or

$$\zeta = o(\varepsilon^{\ell} + q\varepsilon^{u})$$

Despite the physical obscurity of the require-
ment that m be infinite and thus that the horizontal
velocity at the surface be zero, a number of analyses
have been performed using the kinematic condition,
$u(x,o,t) = 0$, as a substitute for the tangential stress
condition. This simplifies the analysis to a considerable
extent, although the interpretation of the tangential
stress condition becomes paradoxical. Since the surface
is presumed to be immobilized, there can be no (first

order) variation in the surface concentrations of the (immiscible) surface active compounds, and so the surface film does not contribute to the balancing of the tangential stress applied by the shearing motion of the viscous liquid beneath the interface. The stress must, however, be balanced somehow, so we are led to the idea that the velocity is not quite zero (so $\xi_s \neq 0$); that the m parameter is not quite infinite; and that the product $(m\xi_s)$ remains of such an order as to balance the viscous stresses. This is a rather unsatisfactory state of affairs and we can readily appreciate Lamb's reluctance to discuss this problem in any detail (see §351).

The m parameter need not be infinite before the surface film begins to exert a pronounced effect on the wave behavior. For example, with q = 0 (the air-water case), the wavenumber, frequency and decay parameters for the two cases under consideration become:

Case 1

$$K_1 = \frac{-m}{(1 + 2\omega_\sigma)(1 - \frac{(1 - i)}{\sqrt{2}} m)} \qquad (6.26)$$

$$\text{Im}\{K_1\} = \frac{m^2}{(1 + 2\omega_\sigma)(1 - \sqrt{2}\,m + m^2)\,\sqrt{2}} \qquad (6.27)$$

Case 2

$$\sigma_1 = \frac{m}{(1 - \frac{1 - i}{\sqrt{2}}\,m)} \qquad (6.28)$$

$$\text{Im}\{\sigma_1\} = -\frac{m^2}{\sqrt{2}\,(1 - \sqrt{2}\,m + m^2)} \qquad (6.29)$$

We can see that $\text{Im}\{K_1\}$ is of the same form as $\text{Im}\{\sigma_1\}$ for gravity waves ($\omega_\sigma = 0$), but it is only 1/3 the magnitude of $\text{Im}\{\sigma_1\}$ for capillary waves ($\omega_\sigma = 1$). These limiting cases are shown in Figure 6-1.

We can also compare the horizontal velocity at the interface with the horizontal velocity of the irrotational wave motion lying just beneath the surface boundary layer. This comparison is most easily made in terms of amplitude and phase response functions as defined below.

FIGURE 6.1

SPATIAL AND TEMPORAL DECAY COEFFICIENTS AS FUNCTIONS OF THE NONDIMENSIONAL ELASTICITY PARAMETER

$I_m\{\sigma_1\}$ and

$I_m\{\kappa_1\}$ for gravity waves

$I_m\{\kappa_1\}$ for capillary waves
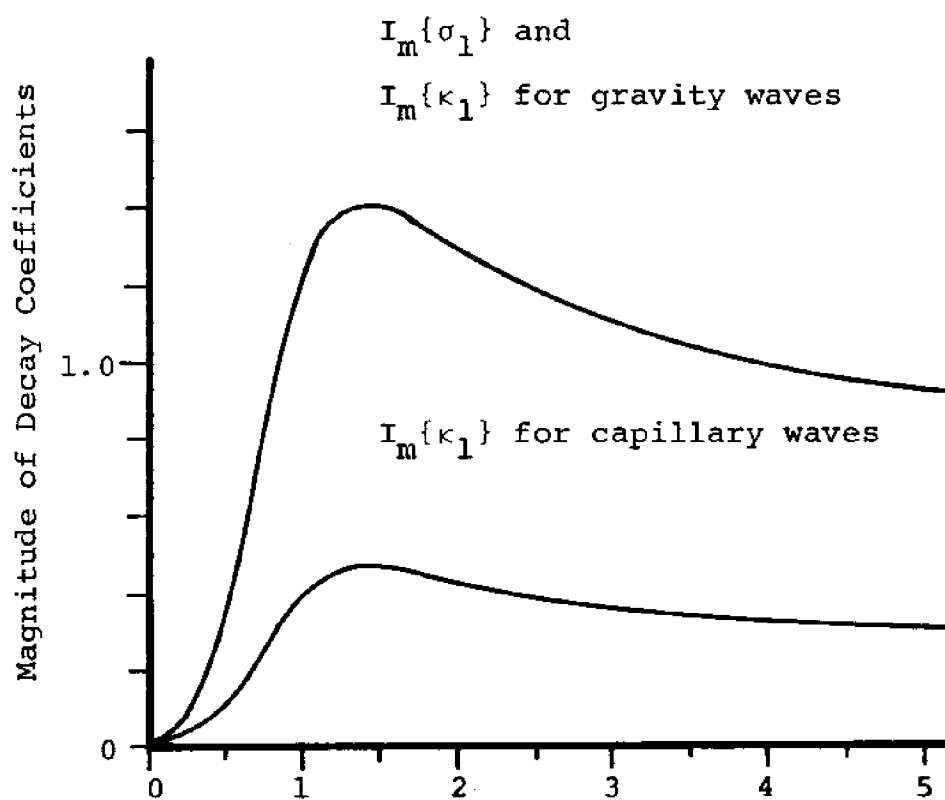
Elasticity Parameter

$$m = \left(\frac{E}{\rho_\ell \delta_\ell C_o{}^2}\right)$$

(nondimensional)

$$A(m) = \left| \frac{u(x,0,t)}{u(x,y_\infty,t)} \right|$$

$$= \left\{ \frac{[Re(b_{10}^\ell - \frac{(1-i)}{\sqrt{2}} c_{10}^\ell)]^2 + [Im(b_{10}^\ell - \frac{(1-i)}{\sqrt{2}} c_{10}^\ell)]^2}{[Re(b_{10}^\ell)]^2 + [Im(b_{10}^\ell)]^2} \right\}^{1/2}$$

$$= [\frac{1}{1 - \sqrt{2}\, m + m^2}]^{1/2} \tag{6.30}$$

$$P(m) = \tan^{-1}\left(\frac{Im[u(x,0,t)]}{Re[u(x,0,t)]} \right) - \tan^{-1}\left(\frac{Im[u(x,y_\infty,t)]}{Re[u(x,y_\infty,t)]} \right)$$

$$= \tan^{-1}\{\frac{-m}{\sqrt{2} - m}\} \tag{6.31}$$

These functions are plotted on Figure 6-2a,b. The locus of points corresponding to the (complex) ratio $\frac{u(x,0,t)}{u(x,y_\infty,t)}$ may be shown to be a circle centered at $(1/2, -1/2i)$ as indicated in Figure 6-c. This circle is parametric in m as indicated, and is analogous to the combined amplitude and phase functions.

These figures show that the effect of the

(a) PHASE RESPONSE EQUATION (6.31)
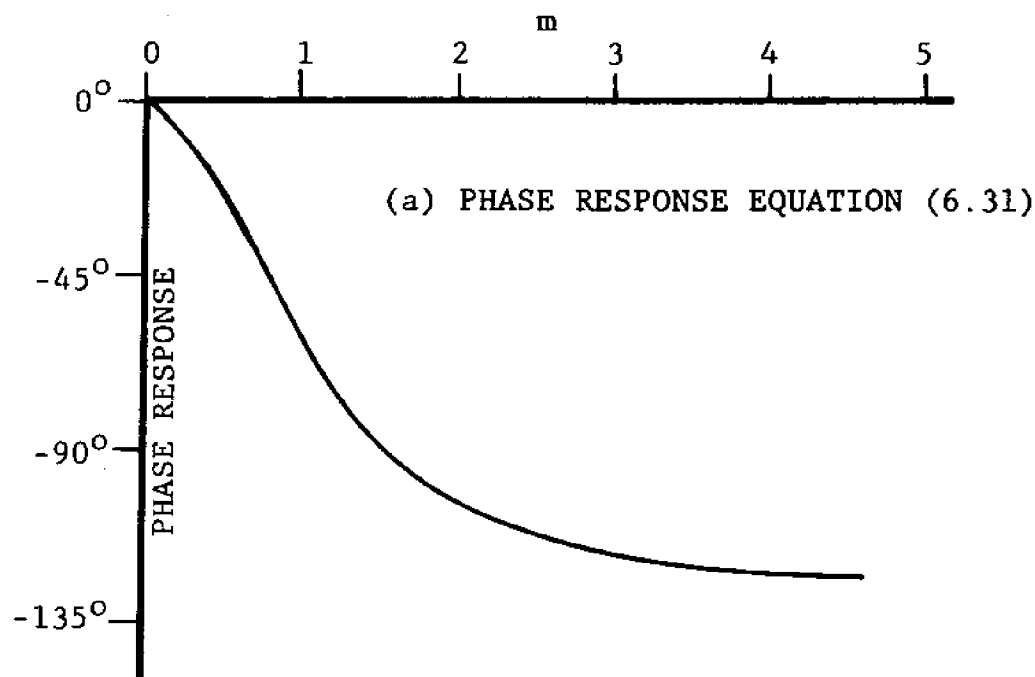
FIGURE 6.2
RESPONSE FUNCTION CHARACTERISTICS

(c) $\dfrac{u(x,o,t)}{u(x,y_\infty,t)}$

(b) AMPLITUDE RESPONSE
EQUATION (6.30)
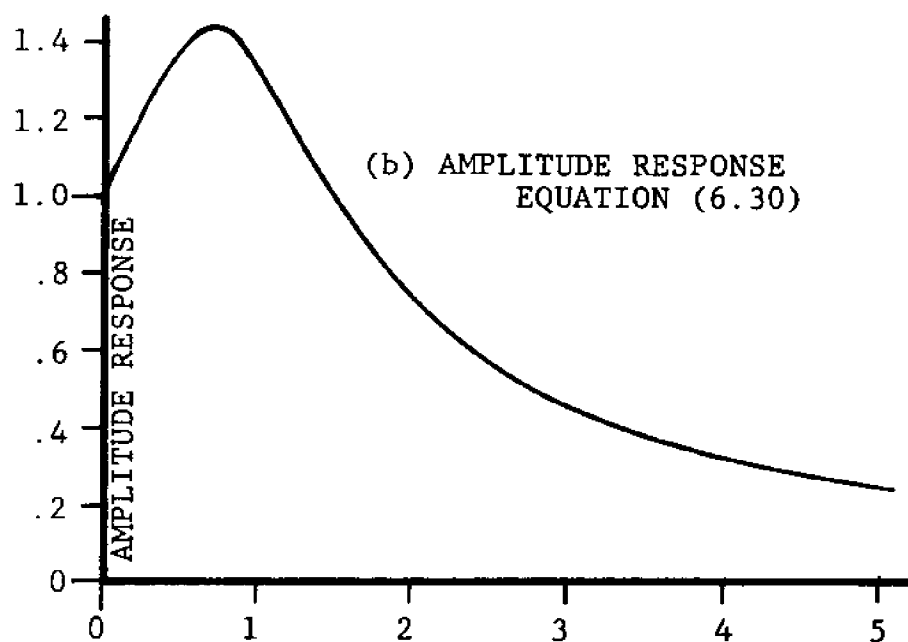
Non-Dimensional Elasticity, m, $(E/(\rho_\ell \gamma_\ell C_o{}^2))$
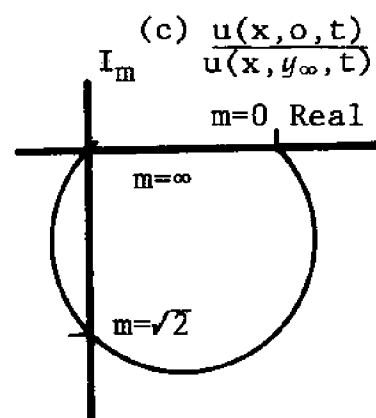
monolayer is pronounced even for values of m considerably
smaller than one. Further, m must reach a value in
excess of 4 or 5 before the amplitude of the interfacial
horizontal velocity is appreciably smaller than the outer,
irrotational velocity.

The extension of this analysis to the more general
case of a soluble surfactant has been attempted by a number
of authors, notably Lucassen (67), Goodrich (61), and
Hansen and Mann (64). The Lucassen paper in particular
attempts to incorporate a more elegant model of the surface
elasticity phenomena. However, and without exception, these
authors found that the only practical way to make headway
was to define an amplitude and phase parameter analogous to
the R and Θ_R of (2.28) and (2.36). At this point the mathe-
matics becomes trivial and the point of difficulty is trans-
ferred to establishing realistic values for R and Θ_R. As
was pointed out in Section 2, there does not appear to be
sufficient experimental data to support this step, and so
our analysis will stop here.

7. FIRST ORDER PROPERTIES OF THIN OIL LAYERS

In the model selected for this study the characteristic depth, d*, for the thick region of an unconstrained oil spill is assumed to be on the order of 1mm. This is in rough correspondence with the Stokes boundary layer depth ($\delta \sim (\nu/\sigma)^{\frac{1}{2}}$) over a range of frequencies for the three characteristic oil types. Table 7.1 presents the frequencies for which the Stokes depth (δ) is 10 times this characteristic depth ($\delta = 10d*$), equal to the characteristic depth ($\delta = d*$), and one tenth the characteristic depth ($\delta = d*/10$) for the three oil types.

TABLE 7.1

Frequency Ranges in which the Stokes
Boundary Layer Depth Correspondes
to Observed Spill Thickness

Generic Oil Type	Frequency RAD/SEC		
	$\sigma(10d*=\delta)$ (Stoke's larger)	$\sigma(d*=\delta)$ (Stoke's equals)	$\sigma(d*/10=\delta)$ (Stoke's smaller)
Light	.03	3	300
Medium	.10	10	1000
Heavy	1.00	100	10,000

It is commonly observed that the peak of the power density spectrum for surface wave amplitude for representative oceanic conditions falls in the range of .5 to 5 radians/second. The spectrum decays rapidly for the frequencies above this peak, the dependence being approximately like ω^{-5} for gravity waves. An approximate upper cutoff frequency beyond which we can reasonably ignore any contributions to the spectrum is thus somewhere in the range of 20 to 50 rad/sec. From the table above, such a limitation on the range of the frequency parameter is seen to insure that the Stokes boundary layer depth is always at least of the same order of magnitude as the spill thickness, and, more typically, the spill thickness is less than or equal to the Stokes depth. This is to say that in the oil spill model we have adopted here, we need only consider non-dimensional oil thicknesses that are similar to, or less than, the non-dimensional boundary layer thickness which in turn suggests we need not include a region of potential (irrotational) flow in the overlying fluid. By the same reasoning, it is also apparent that the depth of the thin regions of the slick are very much smaller than the Stokes depth.

The observation that the thick region may be composed of an aggregate of small lenses of oil is a difficult one to model. One can envisage a variety

of interesting dynamical problems associated with such a
geometry especially if the thin film in-between the
lenses has a moderate elasticity. Unfortunately, we
have no data on either the distribution of lense sizes,
or on the elasticity of the thin film lying between the
lenses.

Moreover, the observational evidence for this
phenomenon is so patchy that there is some possibility
that this is not a general characteristic of unconstrained
oil spills. Consequently, we shall take the option at
this point of modeling the various regions as a
continuous layer of uniform thickness, leaving the
equally interesting lense problem for subsequent
investigation.

From Figure 3.1 we can readily see that the
characteristic horizontal dimension of the thick region
for this particular spill was on the order of 100 m (10^4_{cm}).
With our assumption regarding the continuity of the oil
layer, such large horizontal scales are sufficient to
justify modeling the region as a layer of constant
thickness and infinite extent in the two-dimensional
plane. In fact, we will subsequently find that the
important interaction between the oil and the waves occurs
for wave lengths on the order of 10^2_{cm} or less, and so it

is only necessary to postulate horizontal scales for the
thick region on the order of a few meters to support
this idealization. Figure 7.1 presents the coordinates
for this idealized problem.

The governing equation for the lower fluid
potential region remains the same as in the monolayer
problem discussed above. Likewise, the governing
equations for the oil and water boundary layer regions
remain as developed above. However, it is convenient
in the present problem to modify the functional forms
used in the upper boundary layer so as to take advantage
of the finite depth of the oil layer. We shall therefore
replace the exponential functions of equation 4.24 with
hyperbolic functions, as indicated in equation 7.3 below.
We shall also restrict our discussion to the complex
wavenumber problem (i.e. the spatially decaying wave)
as this is more applicable to an oil spill subjected
to incident waves generated far upwind of the actual
spill. The assymptotic power series expansions are thus:

$$K = 1 + \varepsilon^{\ell} K_{10} + \varepsilon^{u} K_{01} +$$
$$(\varepsilon^{\ell})^2 K_{20} + (\varepsilon^{\ell} \varepsilon^{u}) K_{11} + \ldots; \tag{7.1}$$

$$\sigma = 1; \tag{7.2}$$
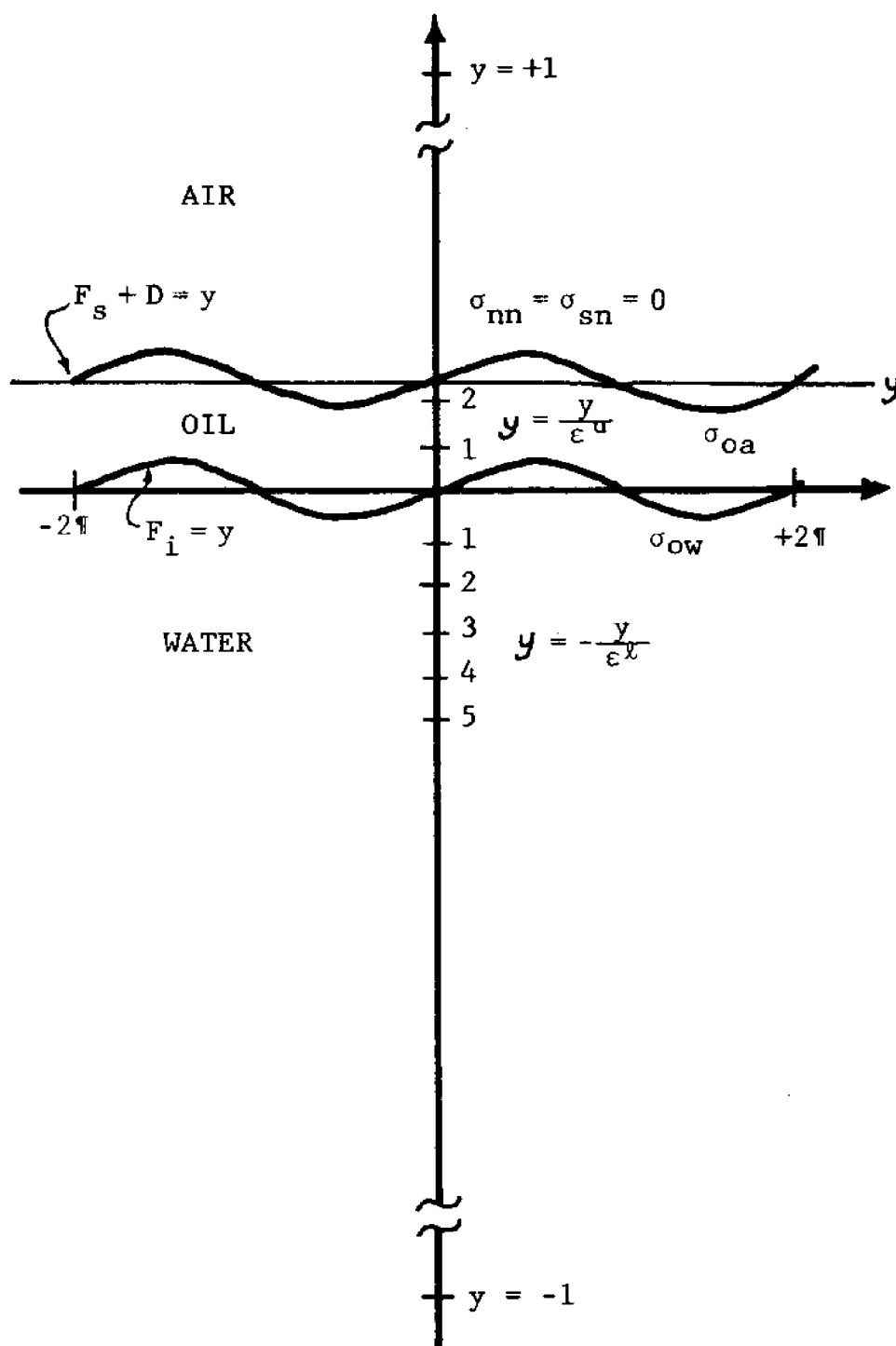
FIGURE 7.1

NON-DIMENSIONAL COORDINATES FOR
THE OIL LAYER PROBLEM

$$\psi_{ij}^u = a_{ij}^u + b_{ij}^u \, y + c_{ij}^u \sinh\left[\frac{(1-i)}{\sqrt{2}}(y-d)\right] +$$

$$d_{ij}^u \cosh\left[\frac{(1-i)}{\sqrt{2}}(y-d)\right],$$

$$\text{for } i = 0,1; \; j = 0,1,2,\ldots; \tag{7.3}$$

$$\psi_{ij}^\ell = a_{ij}^\ell + b_{ij}^\ell \, y + c_{ij}^\ell \, e^{-\frac{(1-i)}{\sqrt{2}} y} + d_{ij}^\ell \, e^{\frac{(1-i)}{\sqrt{2}} y};$$

$$\text{for } i = 0,1; \; j = 0,1,2,\ldots; \tag{7.4}$$

$$\Phi^u = 0 \text{ (no region of purely irrotational flow);} \tag{7.5}$$

$$\Phi^\ell = \text{Re}\{\alpha e^{iKx-it} \, [A_{oo}^\ell + \varepsilon^\ell A_{10}^\ell + \varepsilon^u A_{01}^\ell + \ldots] e^{Ky}\} +$$

$$O(\alpha^2); \tag{7.6}$$

$$\psi^\beta = \text{Re}\{\alpha e^{iKx-it} [\psi_{oo}^\beta + \varepsilon^\beta \psi_{10}^\beta + \varepsilon^i \psi_{01}^\beta + \ldots]\} +$$

$$O(\alpha^2), \tag{7.7}$$

where the ψ_{ij}^β 's are as defined above;

$$F_s(x,t) = \text{Re}\{d e^{iKx-it} [\xi_{oo}^u + \varepsilon^\ell \zeta_{10}^u + \varepsilon^u \zeta_{01}^u + \ldots]\} +$$

$$O(\alpha^2); \text{ and} \tag{7.8}$$

$$F_i(x,t) = \text{Re}\{ae^{iKx-it}[\xi_{oo}^{\ell} + \epsilon^{\ell}\xi_{10}^{\ell} + \epsilon^{u}\xi_{01}^{\ell} + ..]\} +$$

$$O(\alpha^2). \tag{7.9}$$

In the two fluid monolayer problem discussed above we had kinematic matching conditions and Bernoulli's equation to relate the upper potential flow properties to the dynamic and kinematic boundary condition at the interface. In this problem we have kinematic and dynamic boundary conditions to be applied at the surface $y = F_s(x,t) + D$. These can be stated in non-dimensional form using the outer "y" variable as follows:

Kinematic B.C.:

$$\left\{\frac{D}{Dt}(F_s + D - y) = \frac{\partial F_s}{\partial t} + uF_{s_x} - w = 0\right\}_{y=F_s+D} \tag{7.10}$$

Dynamic B.C.s:

  a) Normal Stress:

$$\left\{\frac{\sigma_{nn}^{u}}{\rho_u c_o^2} = -p^u + \omega_g F_s + 2\frac{(1-F_{s_x}^2)}{(1+F_{s_x}^2)}(\epsilon^u)^2 \frac{\partial w}{\partial y} + \right.$$

$$\left. 2(\epsilon^u)^2 \frac{F_{s_x}}{(1+F_{s_x})}\left[\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\right] = \frac{\omega_o^u F_{s_{xx}}}{q(1+F_{s_x}^2)^{3/2}}\right\}_{y=F_s+D} \tag{7.11}$$

b) Tangential Stress:

$$\left\{ \frac{\sigma_{sn}}{\rho_n c_o^2} = (\epsilon^u)^2 \left[\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right] \frac{(1-F_{s_x}^2)}{(1+F_{s_x}^2)} - \right.$$

$$\left. \frac{F_{s_x}}{(1-F_{s_x}^2)} \; 4 \; (\epsilon^u)^2 \; \frac{\partial u}{\partial x} = 0 \right\}_{y=F_s+D} \tag{7.12}$$

We assume here that no monolayer exists at the oil-air
interface thus the zero on the right hand side of equation
(7.12), above. This restriction is imposed solely in the
interest of simplifying the problem. If subsequent
experimental studies either show this interface to have
strong elastic properties or if some discrepancy is found
between the present theory and experiment, this omission
is a likely starting point for subsequent theoretical
developments.

The dynamic and kinematic boundary conditions
at the oil-water interface remain as they were before
(see Section 6).

Kinematic B.C.:

$$\left\{ \frac{D}{Dt} \; (F_i - y) = \frac{\partial F_i}{\partial t} + u F_{i_x} - w = 0 \right\}_{y=F_i} \tag{7.13}$$

Dynamic B.C.'s:

$$\left\{ -p^{\ell} + qp^{u} + \omega_g F_i(1-q) + \right.$$

$$\frac{2(1-F_{i_x}^2)}{(1+F_{i_x}^2)}\left[(\varepsilon^{\ell})^2\frac{\partial w^{\ell}}{\partial y} - q(\varepsilon^u)^2\frac{\partial w^u}{\partial y}\right] +$$

$$2\frac{F_{i_x}}{(1+F_{i_x})}\left\{\varepsilon^{\ell 2}\left[\frac{\partial w^{\ell}}{\partial x} - \frac{1}{\varepsilon^{\ell}}\frac{\partial u^{\ell}}{\partial y}\right] - \right.$$

$$\left. q\varepsilon^{u_2}\left[\frac{\partial w^u}{\partial x} + \frac{1}{\varepsilon^u}\frac{\partial u^{\ell}}{\partial y}\right]\right\} =$$

$$\left.\frac{\omega_{\sigma}F_{i_{xx}}}{(1+F_{i_x}^2)^{3/2}}\right\} \qquad\qquad (7.14)$$

$$y=F_i$$

$$\left\{\left[\frac{1-F_{i_x}^2}{1+F_{i_x}^2}\left[(\varepsilon^{\ell})^2\left[\frac{\partial u}{\partial y} + \frac{\partial \overline{w}}{\partial x}\right] - q(\varepsilon^u)^2\left[\frac{\partial u^u}{\partial y} + \frac{\partial w^u}{\partial x}\right]\right] - \right.\right.$$

$$\left.\frac{F_{i_x}}{(1-F_{i_x}^2)}4\frac{\partial u_i}{\partial x}\left[(\varepsilon^{\ell})^2 - q(\varepsilon^u)^2\right] = \zeta\xi_s\right\} \qquad (7.15)$$

$$y=F_i$$

The non-dimensional parameters used above are defined analogously to those of the monolayer problem discussed in Section 6. The only modifications required by the additional interface are the replacement of ω_σ with ω_σ^i and ω_σ^u, where:

$$\omega_\sigma = \frac{\sigma_{ow} K}{\rho_\ell C_o^2} \; ; \quad \text{and,} \tag{7.16}$$

$$\omega_\sigma^u = \frac{\sigma_{oa} K}{\rho_\ell C_o^2} \; . \tag{7.17}$$

The upper surface capillary parameter, ω_σ^u, is defined in terms of the lower fluid density as it is found that at lowest order (i.e. biggest magnitude), the wave behaves as though the upper fluid weren't present. This convention necessitates the inclusion of the factor q in the denominator of the righthand side of equation (7.11).

The matching condition between the lower boundary layer flow and the outer potential flow also remains unchanged (see equations 5.17 and 5.18).

We may linearize the equations above by expanding about the mean interface levels and thus determine the following $O(\alpha)$ equations

$$w^u = F_{s_t} \qquad (7.18a)$$

$$-p^u + \omega_g F_s + 2(\epsilon^u)^2 \frac{\partial w}{\partial y} = \frac{\omega_\sigma^u}{q} F_{s_{xx}} \qquad (7.18b)$$

$$\frac{\partial u^u}{\partial y} + \frac{\partial w^u}{\partial x} = 0 \qquad (7.18c)$$

$$y=D$$

$$w^u = w^\ell = F_{i_t} \qquad (7.19a)$$

$$-p^\ell + q p^u + \omega_g F_i (1-q) + 2\left[(\epsilon^\ell)^2 \frac{\partial w^\ell}{\partial y} - q(\epsilon^u)^2 \frac{\partial w^u}{\partial y}\right]$$

$$= \omega_\sigma^i F_{i_{xx}} \qquad (7.19b)$$

$$(\epsilon^\ell)^2\left[\frac{\partial u^\ell}{\partial y} + \frac{\partial w^\ell}{\partial x}\right] - q(\epsilon^u)^2\left[\frac{\partial u^u}{\partial y} + \frac{\partial w^u}{\partial x}\right] = \zeta \xi_s \qquad (7.19c)$$

$$y=0$$

It is not immediately obvious from looking at these equations that this specification of boundary conditions differs in an important way from the previous problem's matching and boundary conditions.

However, on reflection, it will be observed

that whereas the matching condition for the monolayer problem between the upper boundary layer flow and the upper potential flow resulted in two equations, one for each of the components of velocity, the present problem has only the one kinematic condition on w, Equation (7.18a). Furthermore, the normal stress condition at y=D, Equation (7.18b), does not help us in its present form because the dynamic pressure is nearly constant throughout the boundary layer region and so it is not affected by variation in u. The tangential stress condition, Equation (7.18c), is useful, as it allows us to determine that $d_{ij}^u = 0$, but this condition had its counterpart in the previous problem where we recognized that exponential growth through the boundary layer was not compatible with the assumed outer flow. Thus we require a condition having to do with the horizontal velocity at y=D so as to close the problem.

This condition is obtained from the normal stress condition by taking the X derivative of Equation (7.18b) and noting that the X derivative of the dynamic pressure is known within the upper boundary layer from the linearized Navier Stokes equation. Thus:

$$\frac{\partial p^u}{\partial x} = -\frac{\partial u^u}{\partial t} + (\varepsilon^u)^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + O(\alpha^2) \qquad (7.20)$$

and

$$- \frac{\partial u^u}{\partial t} + (\varepsilon^u)^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \omega_g F_{s_x} + 2(\varepsilon^u)^2 \frac{\partial^2 w}{\partial x \partial y} - \frac{\omega_\sigma^u}{q} F_{s_{xx}} \qquad (7.21)$$

We can now evaluate the various coefficients by inserting the assumed expansions (being careful to make the proper sign changes in the lower boundary layer) and equating like powers of ε^u, ε^ℓ. Appendix B lists the numerous equations that result. The key variables from our standpoint are the dispersion equations and the coefficients of the highest order rotational flow. These determine the character of the second order flow induced by the surface boundary layer. The dispersion equations are found to be:

$$O(1): \qquad 1 = \omega_g + (\omega_\sigma^i + \omega_\sigma^u) \qquad (7.22)$$

$$O(\varepsilon^u): \qquad K_{01} = \frac{q}{(3 - 2\omega_g)} \{d(1 - \omega_s^2) + \frac{c_{10}^u}{\xi_{00}^u} \sinh \frac{(1-i)}{\sqrt{2}} d \} \qquad (7.23)$$

$$O(\varepsilon^\ell): \qquad K_{10} = \frac{c_{10}^\ell}{\xi_{00}^i (3 - 2\omega_g)} \qquad (7.24)$$

and so,

$$O(\epsilon'): \quad K_1 \equiv K_{10} + QK_{01}$$

$$= \frac{1}{(3 - 2\omega_g)} \{qQd(1 - \omega_s^2) + \frac{(1 + i)}{\sqrt{2}}(1 - \omega_s)$$

$$- \frac{c_{10}^u}{\xi_{00}^i}[\cosh\frac{(1 - i)}{\sqrt{2}}d + \omega_s qQ\sinh\frac{(1 - i)}{\sqrt{2}}d]\} \quad (7.25)$$

where K_1 is the wavenumber perturbation expressed in
terms of the nondimensional water boundary layer depth
(ϵ^ℓ). In deriving these equations, we have made use of
the normal stress condition and the following identities
which may be obtained from the kinematic conditions:

$$O(1): \quad \xi_{00}^u - \xi_{00}^i = 0 \quad (7.26)$$

$$O(\epsilon^\ell): \quad \xi_{10}^u - \xi_{10}^i = 0 \quad (7.27)$$

$$O(\epsilon^u): \quad \xi_{01}^u - \xi_{01}^i = b_{10}^u d + c_{10}^u \sinh\frac{(1 - i)}{\sqrt{2}} d \quad (7.28)$$

We have also made use of the following result which is
obtained from the horizontal gradient of the dynamic
pressure at the upper surface (see discussion above):

$$\frac{b_{10}^{u}}{\xi_{00}^{u}} = \omega_s \tag{7.29}$$

where

$$\omega_s \equiv \omega_g + \frac{\omega_\sigma^{u}}{q} \tag{7.30}$$

From the tangential stress condition and the condition requiring continuity of horizontal velocity at the interface, we have the following equations for the coefficients of the highest order rotational component:

$$c_{10}^{u} = \xi_{00}^{i} \frac{[\omega_s M + \frac{(1+i)}{\sqrt{2}} \frac{(1-\omega_s)}{(1+qQ\tanh[(1-i)d/\sqrt{2}])}]}{[1 - M\frac{(1-i)}{\sqrt{2}}]\cosh\frac{(1-i)}{\sqrt{2}}d} \tag{7.31}$$

and

$$c_{10}^{\ell} = \xi_{00}^{i} \frac{(1+i)}{\sqrt{2}}(1-\omega_s)$$

$$- \xi_{00}^{i} \frac{[\omega_s M + \frac{(1+i)}{\sqrt{2}} \frac{(1-\omega_s)}{(1+qQ\tanh[(1-i)d/\sqrt{2}])}]}{[1 - M\frac{(1-i)}{\sqrt{2}}]} \tag{7.32}$$

where

$$M = \frac{\zeta}{\varepsilon^{\ell} + q\varepsilon^{u} \tanh \frac{(1-i)}{\sqrt{2}} d}$$

$$= m \frac{1}{1 + qQ \tanh \frac{(1-i)}{\sqrt{2}} d} \qquad (7.33)$$

and

m = the Weber number corresponding to
the analogous air-water monolayer
problem, i.e., $m = \zeta/\varepsilon^{\ell}$.

There are a number of similarities between
these results and those obtained for the monolayer problem.
For example, the O(1) dispersion equation [Eq. (7.22)] is
nearly identical to Equation (6.3) when the latter is
specialized for the water-air case ($q \rightarrow 0$) with the excep-
tion that the surface tension parameter is now composed
of the sum of the interfacial and surface parameters
($\omega_{\sigma}^{i} + \omega_{\sigma}^{u}$) instead of the single interfacial parameter,
(ω_{σ}), found in the monolayer problem. At lowest order,
therefore, the motion is like that of a single inviscid

liquid having a surface tension equal to the sum of the two interfacial tensions and a density like that of the underlying fluid.

Equations (7.21) and (7.22) show that the higher order perturbations to the wavenumber (K_{01} and K_{10}) are dependent upon the coefficients of the highest order rotational component of the boundary layer flow (i.e., c_{10}^u and c_{10}^ℓ). Analogous behavior was also found in the monolayer problem [see Equations (6.3b-d) and (6.4b-c)].

This linkage between the C_{10}'s and the $O(\varepsilon)$ perturbations to the wavenumber is a general feature of all free surface wave problems in which the vorticity is stronger than $O(\alpha)$ in the vicinity of the surface. This may be shown as follows.

First, the vorticity may be expressed as a function of the stream functions Ψ^u and Ψ^ℓ as follows [see also Equation (4.8c)]:

$$\omega^\beta = -(\Psi_{yy}^\beta + \Psi_{xx}^\beta) = -(\frac{1}{\varepsilon^{\beta 2}} \Psi_{yy}^\beta + \Psi_{xx}^\beta) \qquad (7.33)$$

Inserting the expansions (7.3), (7.4), and (7.7), we have the following equations for the upper and lower vorticities (ω^u and ω^ℓ respectively):

$$\omega^u = \text{Re}\{\alpha e^{ikx - it}[\frac{1}{\epsilon^u}[-iC_{10}^u \sinh \frac{(1-i)}{\sqrt{2}} (y-d)]]$$

$$+ \frac{\epsilon^\ell}{\epsilon^u} [-iC_{11}^u \sinh[\frac{(1-i)}{\sqrt{2}} (y-d)]] + 0(1)\}$$

$$+ 0(\alpha^2) \qquad\qquad (7.34)$$

$$\omega^\ell = \text{Re}\{\alpha e^{ikx - it}[\frac{1}{\epsilon^\ell}[-iC_{10}^\ell e^{[-(1-i)/\sqrt{2} \ y]}]$$

$$+ \frac{\epsilon^u}{\epsilon^\ell} [-iC_{11}^\ell e^{[-(1-i)/\sqrt{2} \ y]}] + 0(1)\} + \dots \qquad (7.35)$$

Thus, at highest order, the boundary layer vorticity is linearly dependent on the coefficients C_{10}^u and C_{10}^ℓ. Furthermore, its amplitude is $0(\alpha/\epsilon^\beta)$. Also notice that the vorticity at this order is independent of all the other expansion coefficients. Thus, C_{10}^ℓ and C_{10}^u are direct measures of the strength of the lowest order vorticity. To this extent the equations (7.23) and (7.24) thus relate the wavenumber perturbations to the vorticity.

Now it is generally understood that these boundary layer vorticities play an important role in the processes leading to the dissipation of energy in surface waves. It is also commonly known that the energy dissipation leads to complex valued perturbations of the wavenumber (or frequency). Phillips [66], for example, uses the energy dissipation to calculate wave attenuation (see page 37 of Phillips' monograph). However, the degree to which the dissipation of the kinetic and potential energies is tied to the boundary layer vorticity is generally not fully appreciated. Thus, the second observation we shall make relates the energy dissipation explicitly to the boundary layer vorticities. Since the purpose of this discussion is simply to provide additional insight on Equations (7.21) and (7.22), the derivations will be sketched rather briefly, so that we won't wander too far from our central theme which is the exposition of the first order properties of the oil-water system.

Landau and Lifshitz [59] show that the dot (or scalar) product of the momentum equation and the (vector) velocity may be put in the following form (their Equation 16.1, page 53):

$$\frac{\partial}{\partial t}(\frac{1}{2}\rho\,|\underline{u}|^2) + \nabla \cdot \{\rho\underline{u}(\frac{1}{2}|\underline{u}|^2 + p + \rho gy) - \underline{u} \cdot \underline{\sigma}'\}$$

$$= -\sigma'_{iK}\frac{\partial u_i}{\partial x_K} \qquad\qquad (7.36)$$

where $\underline{u}$ is the velocity vector;

$\underline{\sigma}' = [\sigma'_{iK}]$ is the viscous stress tensor;

repeated indices imply summation;

and we have taken the liberty of including a body force
in the bracket on the left hand side which Landau and
Lifshitz ignored in their derivation.

The term on the right hand side of Equation
(7.36) is the rate at which the kinetic and potential
energies of the left hand side are being dissipated into
heat. The term $\underline{u} \cdot \underline{\sigma}'$ corresponds to the work done on the
fluid by the conjunction of the velocity field and the
viscous stresses. It is commonly overlooked.

If we now integrate this equation over an
arbitrarily specified volume V, bounded by the surface Σ,
and invoke the divergence theorem for volume integrals,
we will have the following:

$$\int_V \frac{\partial}{\partial t}(\frac{1}{2}\rho|\underline{u}|^2)dV + \int_\Sigma \underline{n} \cdot \underline{u}(\frac{1}{2}\rho|\underline{u}|^2 + p + \rho gy)d\Sigma - \int_\Sigma \underline{n} \cdot \underline{u} \cdot \underline{\sigma}'\,d\Sigma$$

$$= \int_V \sigma'_{iK} \frac{\partial u_i}{\partial x_K} \, dV \qquad (7.37)$$

where $\underline{n}$ is a unit vector, locally normal to the surface $\sum$, pointing outward.

For a Newtonian, incompressible fluid, we have:

$$\sigma'_{ij} = \mu e_{ij} \equiv \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \qquad (7.38)$$

where e_{ij} is the rate of strain tensor, and μ is the viscosity.

We can readily show that for such an incompressible fluid:

$$\sigma'_{iK} \frac{\partial u_i}{\partial x_K} = \frac{1}{2} \mu \, e_{ij} \cdot e_{ij}$$

$$= \left[2 \left(\frac{\partial u_1}{\partial x_1} \right)^2 + 2 \left(\frac{\partial u_2}{\partial x_2} \right)^2 + 2 \left(\frac{\partial u_3}{\partial x_3} \right)^2 \right.$$

$$\left. + e_{12}^2 + e_{13}^2 + e_{23}^2 \right] \qquad (7.39)$$

Now the off-diagonal rate of strain terms can be readily evaluated in terms of the vorticity as follows:

$$e_{12}^2 = \omega_3^2 + 4 \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \tag{7.40a}$$

$$e_{13}^2 = \omega_2^2 + 4 \frac{\partial u_1}{\partial x_3} \frac{\partial u_3}{\partial x_1} \tag{7.40b}$$

$$e_{23}^2 = \omega_1^2 + 4 \frac{\partial u_2}{\partial x_3} \frac{\partial u_3}{\partial x_2} \tag{7.40c}$$

where $\underline{\omega} = \nabla \times \underline{u} = (\omega_1, \omega_2, \omega_3)$.

Thus, Equation (7.39) may be written:

$$\sigma'_{iK} \frac{\partial u_i}{\partial x_K} = \mu [\omega_1^2 + \omega_2^2 + \omega_3^2 + 4 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}$$

$$+ 2 (\frac{\partial u_1}{\partial x_1})^2 + 2 (\frac{\partial u_2}{\partial x_2})^2 + 2 (\frac{\partial u_3}{\partial x_3})^2] \tag{7.41}$$

But,

$$4 \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + 2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial u_2}{\partial x_2}\right)^2 + 2\left(\frac{\partial u_3}{\partial x_3}\right)^2$$

$$= -2\nabla \cdot \left[\underline{u} \times \underline{\omega} - \frac{1}{2}\nabla \cdot (\underline{u} \cdot \underline{u})\right] \tag{7.42}$$

So,

$$\sigma'_{iK} \frac{\partial u_i}{\partial x_K} = \mu\left[|\underline{\omega}|^2 - 2\nabla \cdot (\underline{u} \times \underline{w}) + \nabla \cdot \nabla(\underline{u} \cdot \underline{u})\right] \tag{7.43}$$

We may also show the following identity which applies to the viscous stress term on the left hand side of Equation (7.37):

$$\underline{u} \cdot \underline{\sigma}' = \mu \nabla |\underline{u}|^2 - \mu \, \underline{u} \times \underline{w} \tag{7.44}$$

Thus, Equation (7.37) becomes:

$$\frac{1}{2}\rho \int_V \frac{\partial}{\partial t}|\underline{u}|^2 dv + \int_\Sigma \underline{n} \cdot \underline{u}\left(\frac{1}{2}\rho |\underline{u}|^2 + p + \rho g y\right) - \mu \int_\Sigma \underline{n} \cdot \nabla |\underline{u}|^2 d\Sigma$$

$$+ \mu \int_\Sigma \underline{n} \cdot \underline{u} \times \underline{\omega} \, d\Sigma = -\mu \int_V |\underline{\omega}|^2 dv - \mu \int_\Sigma \underline{n} \cdot \nabla |\underline{u}|^2 d\Sigma$$

$$+ 2\mu \int_{\Sigma} \underline{n} \cdot \underline{u} x \underline{w} \ d\Sigma \qquad\qquad (7.45)$$

The right hand side of Equation (7.45) corresponds to Lamb's §329 equation (12) upon noting that Lamb's unit normal vector points into the fluid, thus changing the sign of the last two terms. Upon cancelling like terms in Equation (7.45) we have:

$$\frac{1}{2} \rho \int_{V} \frac{\partial}{\partial t} |\underline{u}|^2 \ dV + \int_{\Sigma} \underline{n} \cdot \underline{u} (\frac{1}{2} \rho \ |\underline{y}|^2 + p + \rho g y)$$

$$= -\mu \int_{V} |\omega|^2 dV + \mu \int_{\Sigma} \underline{n} \cdot \underline{u} x \underline{w} \ d\Sigma \qquad\qquad (7.46)$$

In the simplest terms this equation states that the kinetic and potential energies within the volume V are dissipated to heat only when there are regions of vorticity within the fluid. It is apparent that in the absence of vorticity, the work done by the viscous stresses just balances the viscous dissipation. The vorticity thus acts like a catalyst, in that while it doesn't dissipate energy itself (strictly speaking), it does create the conditions

under which the nonviscous mechanical energy terms on the left hand side of Equation (7.37) contribute to the energy dissipation.

We can now see from Equation (7.46) why the wavenumber perturbations depend so strongly upon the coefficients c_{10}^{u} and c_{10}^{ℓ}. In the problem at hand, the assumption that the waves have a purely sinusoidal time dependence suggests that if energy is dissipated within the fluid, then the mean-squared wave amplitude (which is an accurate measure of the kinetic and potential energies of the lightly damped waves) must decrease in the direction of energy propagation. Clearly, if the dissipation is large (implying substantial wave amplitude attenuation), then the imaginary wavenumber components will also be large. From the right hand side of Equation (7.46) we can see that large dissipation corresponds to having regions of strong mean-squared vorticity within the fluid, and small positive, or even negative, values of the time averaged surface integral. Since the lowest order vorticity is linearly related to the c_{10}^{u} and c_{10}^{ℓ} coefficients [Equations (7.34) and (7.35)], this explains the close relationship between the wavenumber perturbation and these coefficients. Equation (7.33) suggests that subsequent terms in the wavenumber expansion will exhibit

much more complex dependencies upon the stream function coefficients due to the increasing number of terms that will enter into the vorticity expansion.

We also find that the wavenumber perturbations, Equations (7.23) and (7.24), depend upon the parameter ω_s which is defined by Equation (7.28). This parameter is seen to be analogous to the right hand side of the inviscid dispersion equation for a single fluid system having the density of the upper layer and a surface tension equal to the air-oil surface tension. Thus $(1 - \omega_s)$ and $(1 - \omega_s^2)$ are measures of the difference between the irrotational flow that would exist at the specified frequency if the system were made up of a single fluid having the properties of a very deep upper layer as compared to the flow properties that actually exist. For pure gravity waves, ω_s equals unity (irrespective of interfacial elasticities) and the wavenumber perturbations rely soley on C_{10}^u and C_{10}^ℓ.

This behavior can be explained rather simply as follows. In the event that ω_s equals unity, the irrotational flow would satisfy the condition of zero slip at the oil-water interface. In this case the rotational flow established at the interface would have the task of satisfying the continuity of tangential stress condition

and nothing more. However, to the extent that ω_s is not equal to unity, then the upper and lower fluids would, except for viscosity, oscillate in such a fashion as to create a slippage at the interface. In this case, the interfacial boundary layer would have to provide a perturbation to the irrotational flow that allows both continuity of tangential velocity and tangential stress. Thus, regions of strong voriticity could be formed at the interface of two fluids even if their densities and viscosities were equal, provided only that ω_s was different from unity. Lamb commented on this slippage phenomena as it applies to gravity waves in §231 in which he noted that for deep water waves in two superposed fluids, the slippage condition was satisfied.

The equations as they are shown above are far too cumbersome to provide a detailed insight on the various dependencies. It is helpful, therefore, to specialize the equations for a few limiting cases. First, writing K_1 in terms of the known forms of C_{10}^u and C_{10}^ℓ, we have:

$$K_1 = \frac{1}{(3 - 2\omega_g)} \{ qQd(1 - \omega_s^2) + (1 - \omega_s) \frac{(1 + i)}{\sqrt{2}}$$

$$- \frac{[\omega_s M + \frac{(1 + i)}{\sqrt{2}} \frac{(1 - \omega_s)}{(1 + qQ \tanh [(1 - i)d/\sqrt{2}]}]}{[1 - M\frac{(1 - i)}{\sqrt{2}}]}$$

$$* [1 + qQ\omega_s \tanh \frac{(1 - i)}{\sqrt{2}} d] \} \qquad (7.47)$$

Expanding this about d=0, we have:

$$K_1 = \frac{-M}{(3 - 2\omega_g)(1 - \frac{(1 - i)}{\sqrt{2}} M)} + \frac{qQd}{(3 - 2\omega_g)}$$

$$\cdot \frac{[2(1 - \omega_s) - M\omega_s^2 \frac{(1 - i)}{\sqrt{2}}]}{[1 - \frac{(1 - i)}{\sqrt{2}} M]} + O(d^2) \qquad (7.48)$$

At lowest order, this is simply the monolayer equation. Further, K_1 is seen to be pure real through $O(d)$ if $M = 0$. That is, if the interface between the two fluids has a negligible elasticity, then no damping will occur at this order. This has some obvious applications to the oil spill model for it suggests that once sufficient surfactants have

been extracted from the thin region, where $d = O(10^{-3})$, then this region will no longer interact strongly with the wave field.

Considering the other extreme, we can take d to be large enough so that

$$\tanh \frac{(1-i)}{\sqrt{2}} d \simeq 1.0$$

(e.g., d=5). The equation for K_1 now becomes:

$$K_1 = \frac{1}{(3 - 2\omega_g)} \left\{ qQd(1 - \omega_s^2) \right.$$

$$+ \frac{qQ(\frac{1+i}{\sqrt{2}})(1 - \omega_s)^2}{(1 + qQ)(1 - M\frac{(1-i)}{\sqrt{2}})}$$

$$\left. - \frac{M(1 + \omega_s^2 qQ)}{(1 - M\frac{(1-i)}{\sqrt{2}})} \right\} \tag{7.49}$$

Here we can see that the effect of the thickness is to both multiply the thin film monolayer behavior [Equation (7.48)] by a factor $(1 + \omega_s^2 qQ)$ and to give rise to the two other terms in the bracket. We can readily see that if ω_s is similar to unity, then these other two terms

are small, and the predominant effect is to amplify the monolayer properties.

The linear dependence upon d exhibited in the first term in the bracket is an artifice of the analysis to the extent that the model assumes that the upper fluid is not thick enough to support a predominantly irrotational core region. As d becomes substantially larger than one, this assumption becomes less and less valid; and the necessity for including an upper region of potential flow grows. One important effect a region of irrotational flow in the oil would have would be to modify the dispersion equations [(7.20) to (7.23)]. It is most likely that the present analysis is accurate only up to values of d similar to 5 or so.

Although these equations are considerably simplified by the assumption regarding d, it is clear that the complex interdependencies require some concrete examples if we are to learn anything more about these first order properties. In particular, we require some knowledge of the ω_s and $\text{Im}\{K_1\}$ behavior over a range of frequencies. Figure 7.2 depicts ω_s for the three classes of oil over a range of frequencies sufficient to cover the dynamically important constituents of the surface wave field. Our assumptions regarding the interfacial tension
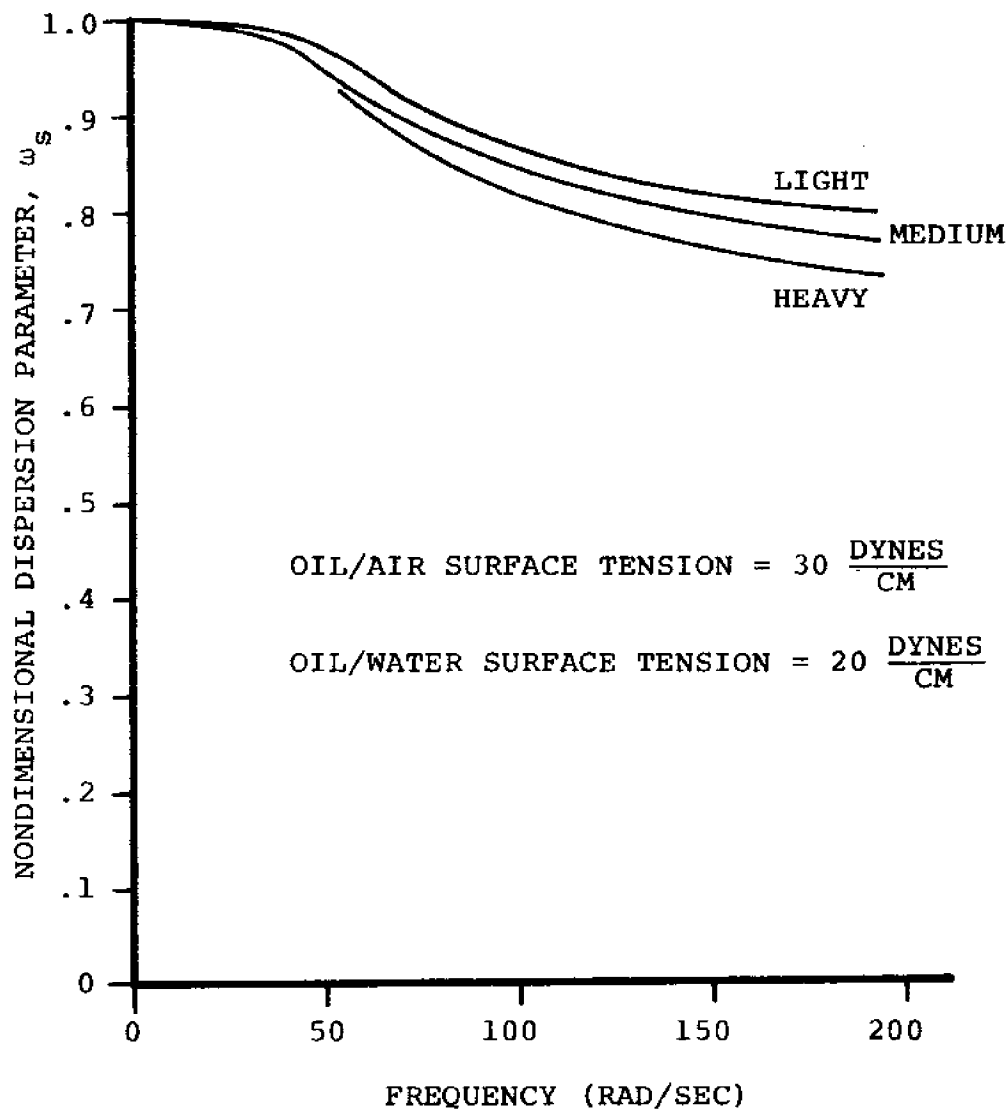
FIGURE 7.2

VARIATION OF ω_s WITH FREQUENCY
FOR THREE GENERIC OILS*

*SEE TABLE 2.1

(σ_{ow} = 20 dynes/cm) and the surface tension (σ_{oa} = 30 dynes/cm) may be regarded as conservative insofar as ω_s is concerned since the effect of lower tensions (the likely prospect) would be to increase the relative amplitude of ω_g which would increase the frequency at which the ω_s curve departs significantly from unity.

Corresponding to these values of ω_s, are the thickness related decay lengths depicted on Figure 7.3 (m is assumed to be zero in these calcuations). The decay length is defined as the distance a wave must travel in order that its amplitude be diminished by a factor e^{-1} (= .368). In this distance the wave's energy will have been attenuated by the factor e^{-2} (= .135), and so this is a convenient scale with which to characterize the horizontal decay of the wave. This distance is readily calculated from the formula:

$$\lambda_1 = \frac{1}{K_0 \epsilon \, I_m\{K_1\}} ,$$

(7.50)

where K_0 is the appropriate root of the dispersion equation, equation (7.20), which has the following dimensional form:

FIGURE 7.3

DECAY LENGTHS FOR THREE GENERIC OILS
ASSUMING SPILL IS "THICK" AND DAMPING IS BY THICKNESS ONLY

$$\lambda_1 = \frac{1}{K_o \varepsilon I_m \{K_1\}}$$
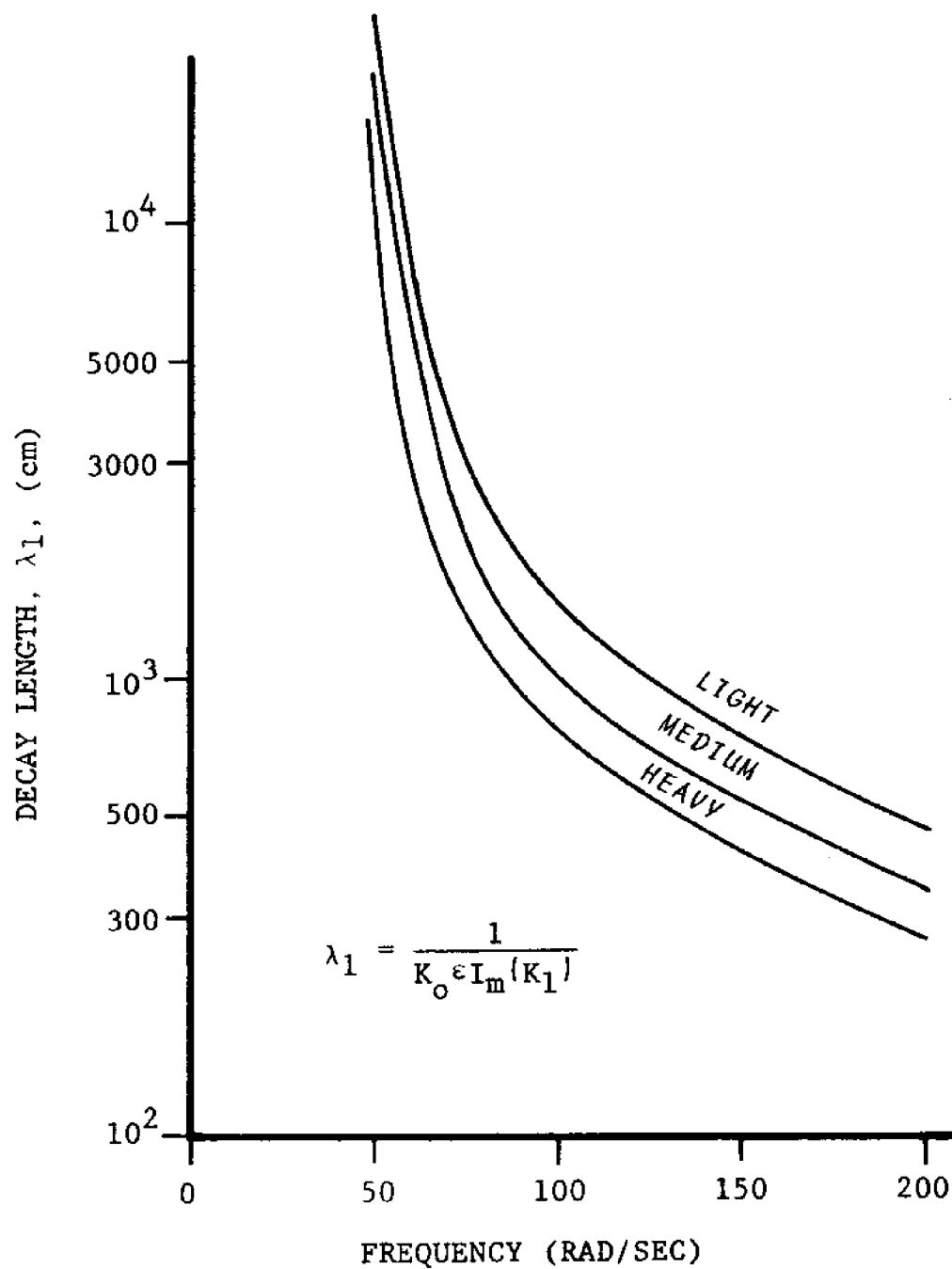
$$1 = \frac{g K_o}{\sigma^2} + \frac{\sigma_{oa} K_o}{\rho_\ell \sigma^2} + \frac{\sigma_{ow} K_o}{\rho_\ell \sigma^2},$$ (7.51)

(σ is the wave frequency in radians per second).

If we now consider the combined effects of monolayer elasticity and thickness we can see that for moderate values of m the decay lengths become much smaller. Figure 7.4 shows λ_1 for a thick oil layer assuming m is pure real (i.e., for thick oil this implies that the diffusion induced phase lag in the surface concentration of surfactants is negligible) and of value .5. Medium oil densities and viscosities are used. Notice that for frequencies in the range of 10 - 50 radians/second, the characteristic decay length, λ_1, is in the range 50 cm to 1000 cm. The corresponding values of the surface compressional modulus associated with an M of .5 lie in the range of E = 300 dynes/cm at 10 radians/second, to about E = 10 dynes/cm at a frequency of 50 radians/second. At the lower frequencies (i.e. 10 radians/second) such values for the elasticity are high but even here we cannot rule them completely implausible.

Holding M constant while varying the thickness (a physically difficult experiment no doubt due to the

FIGURE 7.4

DECAY LENGTH AS A FUNCTION OF FREQUENCY
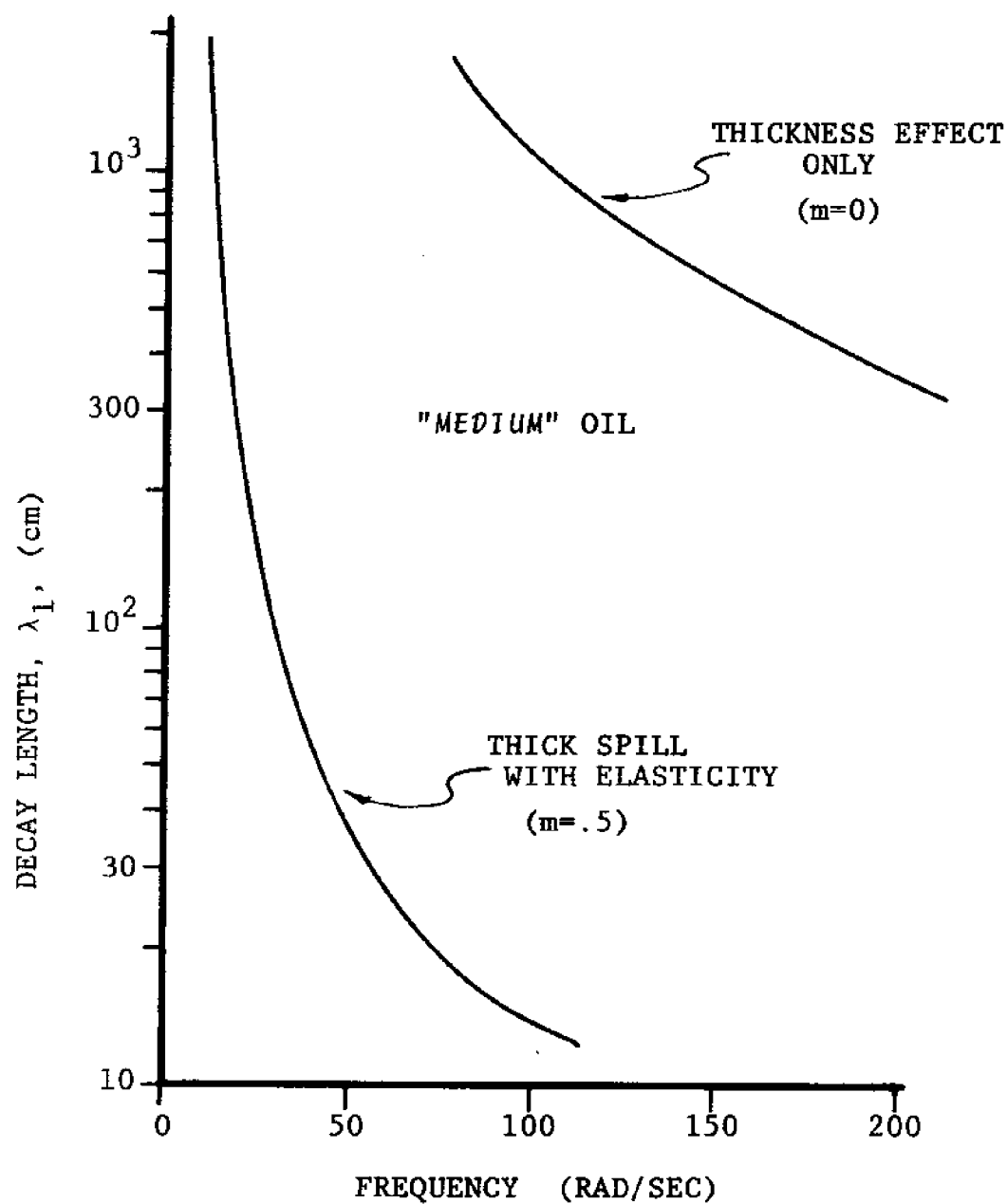
complex 'd' dependency of M) we can see from Figure 7.5
that the thickness effect is not monotonic. Rather
the value of $Im(K_1)$ reaches a peak in the vicinity of
d = 1, and then tapers rapidly to its asymptotic
value which in this case is 1.098.

Up to this point, we have implicitly assumed
that the x variations in spill thickness and elasticity
were negligible. The latter assumption can be argued
from physical grounds as follows. First, it is reasonable
to assume that the bulk of the surfactants contributing
to the interfacial elasticity will come from the spilled
oil, at least initially.* Under these circumstances, we
might expect the distribution of surfactants to be
reasonably uniform over fairly large length scales
within the thick region of the thicker oil layer.
Since the surface elasticity is a function of surface
tension, and since the system will naturally tend to
a state in which the surface tension is uniform over
the interface, we should expect the elasticity to be
reasonably uniform over moderately large length scales.

---

*Over long time periods, we might speculate that biological
agents metabolizing selected petroleum hydrocarbons will
probably be the principal source of surface active
compounds.

FIGURE 7.5

$I_m\{K_1\}$ VERSUS NON-DIMENSIONAL SPILL THICKNESS
FOR FIXED M

NON-DIMENSIONAL SPILL THICKNESS
$(d = \varepsilon^{-1} DK_o)$
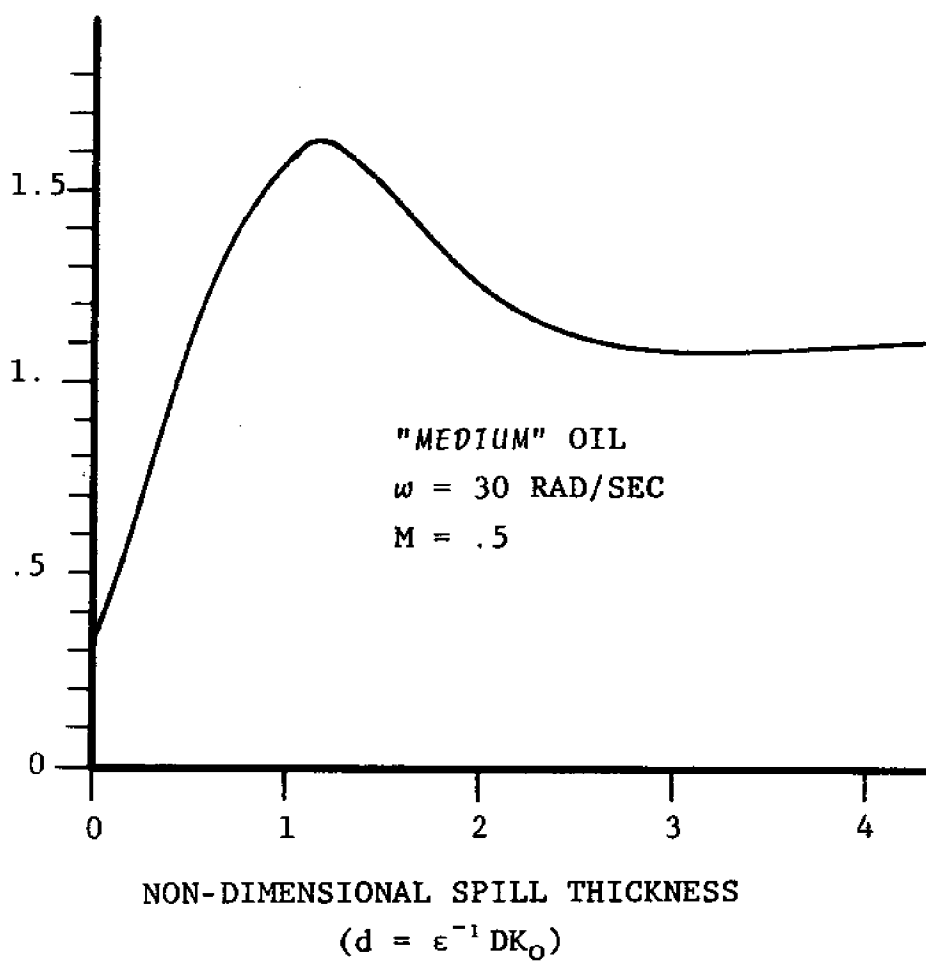
The principal objection to this generalization being that spatial variations in the differential extraction of the surfactants due to local irregularities in the air and water flow fields may lead to spatially heterogeneous mixtures of surfactants that exhibit similar surface tension but dissimilar elasticities. Nevertheless, a first approximation would be that the interfacial elasticity is reasonably uniform.

However, in Section 3 we indicated that the thickness of the thick larger may vary by plus or minus a factor of two. Thus, an interesting question with direct physical application is how does the imaginary portion of the wavenumber perturbation vary with depth for constant elasticities, and what effect might this have on the present analysis. Figure 7.6 shows the variation of $Im\{K_1\}$ with depth for three values of elasticity for a frequency of 30 radians/second and a medium oil. The elasticities are assumed to be pure real, thus we ignore possible phase lags associated with diffusion related relaxation. Notice that for a very weak surface elasticity ($E = 3$ dynes/cm), the wavenumber perturbation is negligible, and monotonic decreasing. However, for the two other cases studied,

FIGURE 7.6

$I_m(K_1)$ VERSUS SPILL THICKNESS FOR
THREE ELASTICITIES

SPILL THICKNESS (mm)
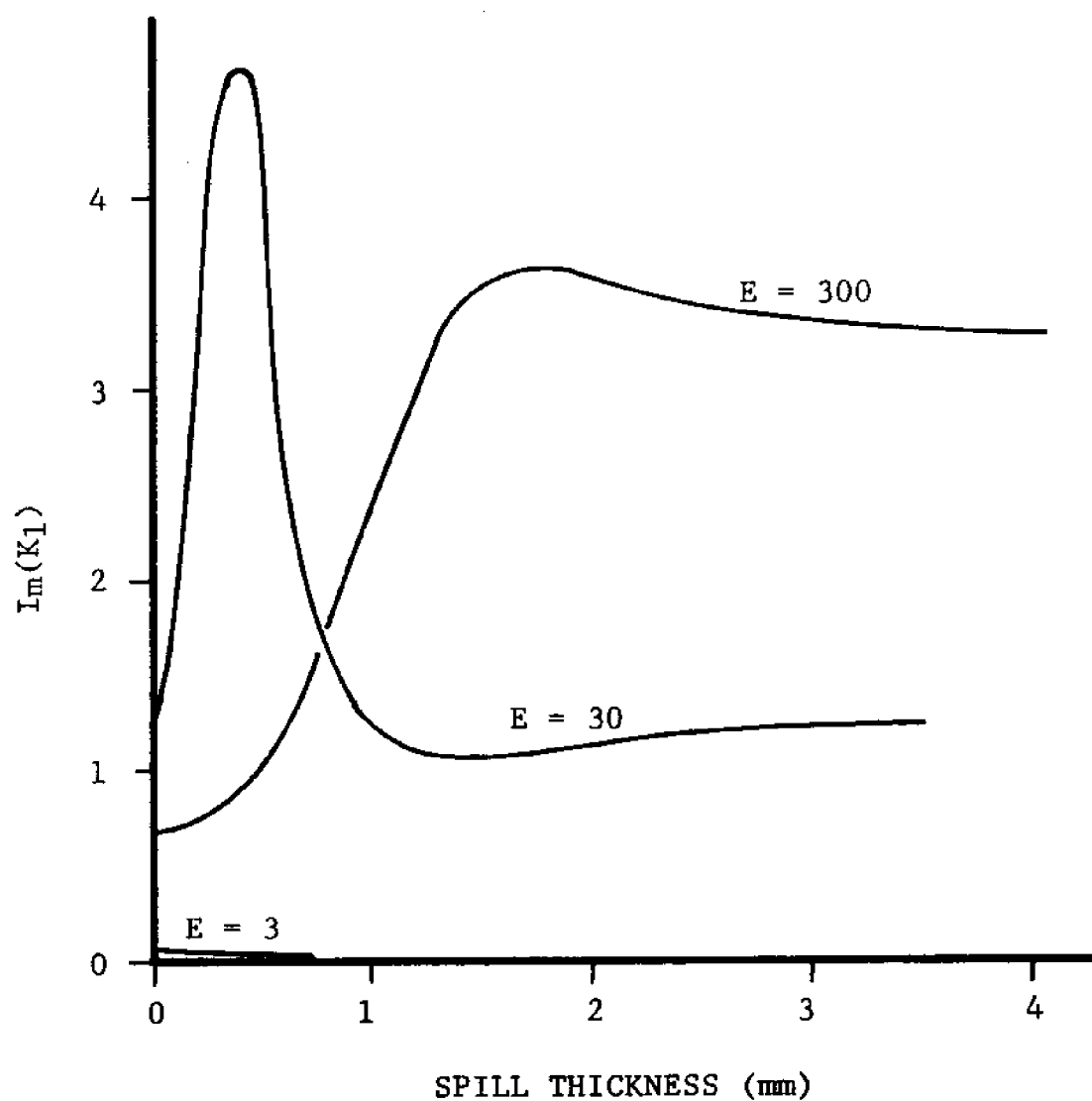
"MEDIUM" OIL

FREQUENCY = 30 RAD/SEC

WAVELENGTH = 7.1 cm

E = 30 dynes/cm, and E = 300 dynes/cm, the behavior is quite interesting. For an elasticity of 30 dynes/cm we find a very sharp peak in $\text{Im}\{K_l\}$ at a (dimensional) depth of about .2 mm (for the oil and frequency specified). If we define amplitude and phase response functions for the water phase in an analagous fashion to those of Section 6 we find that at this depth, the water at the interface has a horizontal velocity that is 2.7 times the amplitude of the horizontal velocity just beneath the Stokes layer. The interfacial horizontal velocity is also found to lag its irrotational counterpoint by about 70 degrees. Thus, we find that the oil layer contributes to a strong resonance at the oil water interface. This in turn induces a greatly amplified vorticity which leads to the enhanced damping.

The curve for the higher elasticity, E = 300 dynes/cm, exhibits similar features, but now the wave frequency is too low for the system to be at resonance, and the ocillations at the interface are greatly attenuated. In fact, at the peak in the $\text{Im}\{K_l\}$ curve, the amplitude response is found to be .25 and the phase response is -127°. Thus, the response is somewhat analogous to that of a flexible but incompressible sheet on the surface of the water (see Section 6). However,

since the flexible sheet approximation does not model the contribution of the oil layer to the dissipation of wave energy, the decay coefficient for the flexible sheet is too low. In the case of a surface elasticity of 300 dynes/cm, and a medium oil, the error is about a factor of five at 30 radians/second.

The implications of this depth dependence are quite important to the problem at hand. In particular, if the average spill depth coincides with the resonance depth, then small variations in depth about the mean will induce large variations in the local wave decay parameter. Such a possibility was not provided for in the initial formulation of the problem in which we considered the principal horizontal scale to be the wavelength. Therefore, in such regions, our present results can only be considered as rough approximations pointing the way for further analysis. Furthermore, when $\text{Im}\{K_1\} \gg 1$, as in Figure 7.6, we must begin to wonder if the εK_1 terms are still negligible compared to 1.

To appreciate the full magnitude of this situation it is useful to plot the ratio of the decay length, λ_1, divided by the wavelength, λ_0, against the oil layer depth for the medium oil with an

interfacial elasticity of 30 dynes/cm for a number of
frequencies around 30 radians/second. As we can
see from Figure 7.7, for small depths and at the
frequencies shown, the ratio is initially of a fairly
high value, say 10, but it thereupon decreases with
increasing depth. Depending upon the frequency, the
curves reach some minimum value and then rise to an
asymptote. For frequencies below 40 radians/second
the asymptotic value is high enough so that for thick
layers the λ_1 scale can probably be ignored. However,
at 40 radians/second we can see that the assymptotic
value of the ratio is only about 2.5:1, so we must
consider the theory suspect in this range.

Another feature of the depth dependence of the
ratio (λ_1/λ_o) for the constant elasticity case is
that there is a lower frequency below which λ_1/λ_o
grows monotonically with increasing depth as opposed
to the behavior depicted above. In Figure 7.7, for
example, if the vertical axis were extended to a value
of 200, then the ratio (λ_1/λ_o) for a frequency of
20 radians/second would begin at about 30 and rise
asymptotically to a value of 178. Since increasing
λ_1/λ_o corresponds to decreasing $\text{Im}\{K_1\}$, this critical
frequency can be found by equating the derivative with

FIGURE 7.7

ω	λ_0 (cm)	δ_0 (mm)	ε_0	ε_w
25	10.07	.48	.03	.009
30	7.13	.44	.04	.01
40	4.28	.38	.06	.02

Oil Properties

σ_{oa} = 30 dynes/cm

σ_{ow} = 20 dynes/cm

E = 30 dynes/cm

ρ = .85 gm/(cm)^3

μ = 10 cp.

(y-axis) NUMBER OF WAVELENGTHS IN A DECAY LENGTH

$\lambda_1 \div \lambda_0$

25 rad/sec

30 rad/sec

40 rad/sec

(mm)

THICKNESS OF OIL LAYER
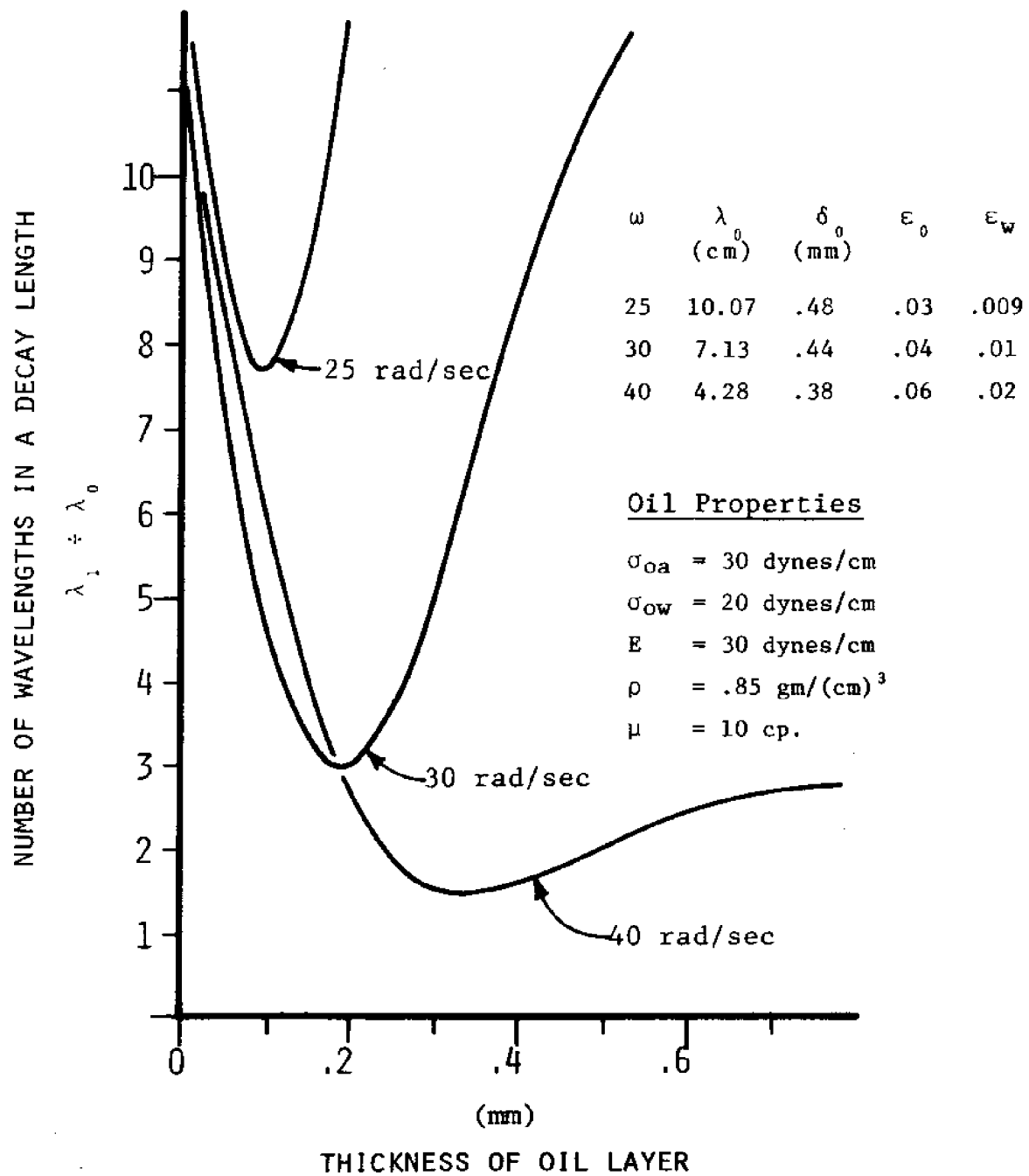
depth of the imaginary component of equation (7.48) to zero and evaluating at d = 0. The resulting equation is fourth order in the parameter M. Since M is dependent upon the square root of the frequency, the determination of the proper root, M^*, suffices to establish the critical frequency.

This calculation has not been made, although it would be rather straightforward. However, the essential fact has already been established using Figure 7.7.

The implications of this observation are rather important if we consider a spectral desciption of the ambient wave field and the essentially linear characteristic of the present analysis (thus allowing superposition). At the up wind edge of the slick the various spectral components of the wave field will pass from a region of thin oil into the thick region. In thus passing through a range of oil depths, the waves above the cutoff frequency will, at some point, interact vigorously with the oil. Once we pass more than a few wavelengths into the slick, these components of the spectrum will be nearly entirely dissipated, assuming the present theory is at least qualitatively correct.

Meanwhile, the lower frequency components will pass only slightly attenuated into the thick region of the spill. Thus, if we are to gain a detailed understanding of the spill's interaction with waves, we require a more complete understanding of the interactions that might occur in the edge regions. The analysis to support such an investigation does not appear to be a simple one, however, and we leave it for subsequent writers.

Figure 7.8 shows the decay length, λ_1, for a range of elasticities assuming a uniformly deep oil (eq. d $\simeq$ 5) and neglecting the edge effects (i.e. the depth dependent resonance discussed above). Notice that the decay length for an elasticity of 30 dynes/cm is shorter than that of the 300 dynes/cm case, over the range of frequencies lying above 36 radians/second. This behavior is analogous to that observed for the monolayer in which it was observed that for values of M similar to unity the damping was a maximum. In the present case, the higher elasticity leads to a larger M, which has the effect of reducing the wave damping once M is larger than 1 or 2.

FIGURE 7.8

DECAY LENGTH AND WAVELENGTH AS A FUNCTION OF FREQUENCY FOR A "THICK" LAYER OF MEDIUM OIL
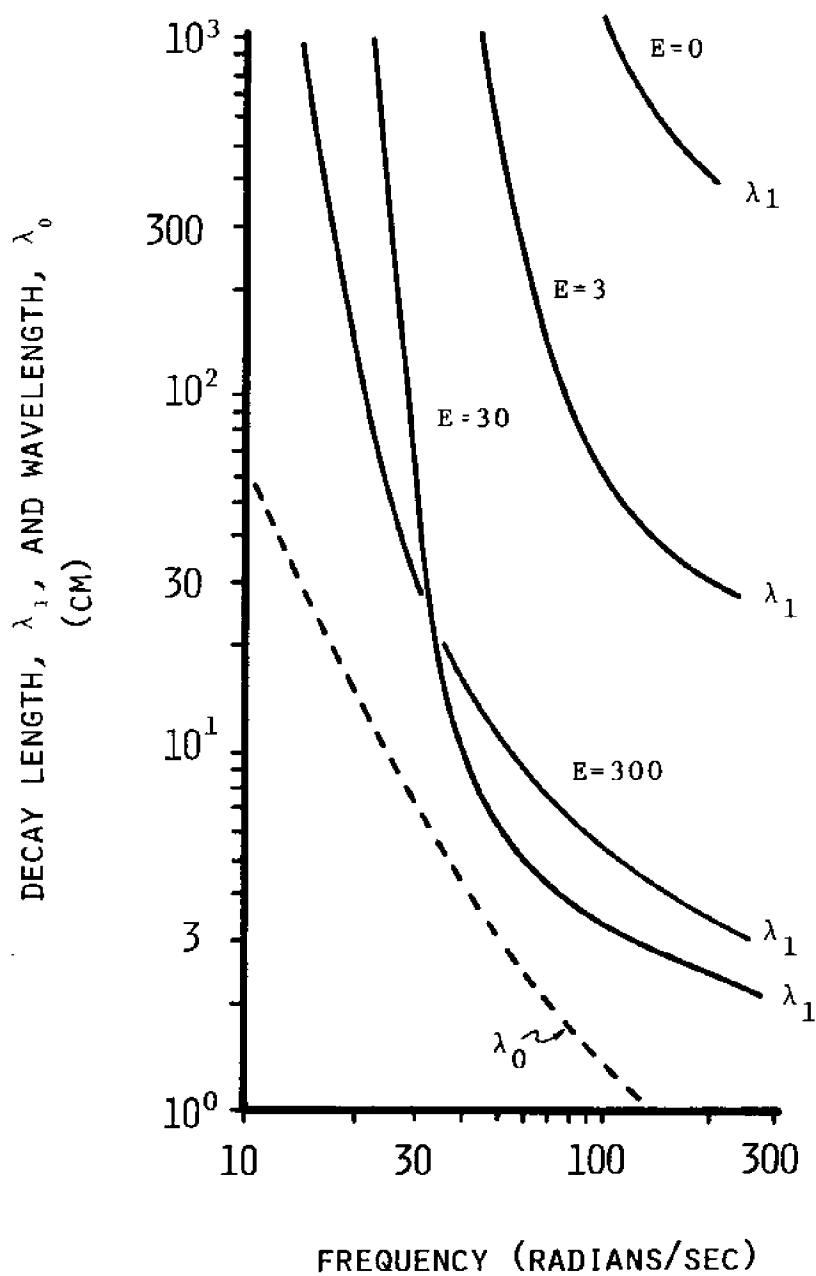
8. SECOND ORDER STOKES BOUNDARY LAYER FLOW

In the absence of strong secondary boundary layer flows the order ϵ^1 terms in the wavenumber expansion (see Equation 8.23 for example) play an important role in determining the character of the second order flow beneath an oil slick subjected to monochromatic waves. This can best be shown from fairly general considerations of the horizontal and vertical momentum equations.

We begin with the nondimensional horizontal and vertical momentum equations expressed as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \epsilon^2\nabla^2 u \qquad (8.1)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y} = -\frac{\partial p}{\partial y} - \omega_g + \epsilon^2\nabla^2 w \qquad (8.2)$$

Here we have used the characteristic wave scales in the nondimensionalization process. For the present we shall ignore the possible presence of an oil layer on the surface, treating the material in the subsurface region as a homogeneous, incompressible, Newtonian fluid. We impose the condition of zero normal stress at the surface $y = \xi$. The approach will be to integrate the vertical momentum equation and take a time average so as to establish the

value of both the time dependent and time steady pressure beneath the surface. We shall then integrate the horizontal momentum equation and take a time average. Assuming that the vertical variation of the steady second order velocity is much greater than the horizontal variation $(\partial^2\bar{u}/\partial y^2 \gg \partial^2\bar{u}/\partial x^2)$, we can then establish relationships for the form of $\partial\bar{u}/\partial y$ at the lower edge of the Stokes boundary layer.

The technique used here for ordering and evaluating the integrals follows from two papers by Longuet-Higgins and Stewart [62,64]. The integration of the vertical momentum equation follows [62] and the ordering of the integrated horizontal momentum equation is somewhat like the [64] paper.

Rearranging the vertical momentum equation (8.2) and utilizing the following identity which springs from the continuity equation,

$$\frac{\partial}{\partial x}\{y\frac{\partial u}{\partial t}\} + \frac{\partial}{\partial y}\{y\frac{\partial w}{\partial t}\} = \frac{\partial}{\partial t}\,y\{\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}\}$$

$$+ \frac{\partial w}{\partial t} = \frac{\partial w}{\partial t}$$

we have:

$$- \frac{\partial p}{\partial y} = \omega_g + \frac{\partial}{\partial y} \{y \frac{\partial w}{\partial t} + w^2\}$$

$$+ \frac{\partial}{\partial x} \{y \frac{\partial u}{\partial t} + uw\} - \epsilon^2 \nabla^2 w \qquad (8.3)$$

Integrating this equation from a point $y = -y*$, to the surface at $y = \xi$, we have:

$$p(-y*) = p(\xi) + \omega_g(\xi + y*) + \{y \frac{\partial w}{\partial t} + w^2\}_{-y*}^{\xi}$$

$$+ \int_{-y*}^{\xi} \{\frac{\partial}{\partial x} [y \frac{\partial u}{\partial t} + uw]\} - \epsilon^2 \int_{-y*}^{\xi} \nabla^2 w dy \qquad (8.4)$$

It should be noted that this is an exact result.

The surface kinematic condition is given by:

$$\{\xi + u\xi_x = w\}_{y=\xi} \qquad (8.5)$$

Thus, we can evaluate the first bracketed term on the right hand side of Equation (8.4) at the upper limit, $y = \xi$, as follows:

$$\{y \frac{\partial w}{\partial t} + w^2\}_{y=\xi} = \xi[\ddot{\xi} + \frac{\partial}{\partial t}(u\xi_x)] + (\dot{\xi} + u\xi_x)^2$$

$$= \xi\ddot{\xi} + \dot{\xi}^2 + \xi[\dot{u}\xi_x + u\dot{\xi}_x]$$

$$+ 2u\dot{\xi}\xi_x + u^2\xi_x^2$$

$$= \frac{1}{2} \frac{\partial^2}{\partial t^2} (\xi^2) + O(\alpha^3) \qquad (8.6)$$

where $\frac{\partial}{\partial t} (\ ) = (\dot{\ })$.

The normal and tangential stress conditions at fluid interfaces may be expressed in terms of the rate of strain tensor components and the pressure. For two fluids, the equations are:

$$-p^\ell + qp^u + \frac{(1 - \xi_x^2)}{(1 + \xi_x^2)} (\varepsilon^{\ell 2} e_{yy}^\ell - q\varepsilon u^2 e_{yy}^u)$$

$$- 2 \frac{\xi_x}{(1 + \xi_x^2)} (\varepsilon^{\ell 2} e_{xy}^\ell - q\varepsilon u^2 e_{xy}^u) = \frac{\xi_{xx}\omega_\sigma}{(1 + \xi_x^2)^{3/2}} \quad ; \quad (8.7)$$

$$\text{at } y=\xi;$$

and,

$$
\frac{(1-\xi_x^2)}{(1+\xi_x^2)} \left[\varepsilon^{\ell^2} e_{xy}^{\ell} - q\varepsilon^{u^2} e_{xy}^u \right]
$$

$$
+ 2 \frac{\xi_x}{(1+\xi_x^2)} \left[\varepsilon^{\ell^2} e_{yy}^{\ell} - q\varepsilon^{u^2} e_{yy}^u \right]
$$

$$
= \zeta\xi_s ; \qquad \text{at } [y=\xi] ; \qquad\qquad\qquad (8.8)
$$

where the quantities are as defined in the discussion of the first order problem (see page   ). For the present discussion we may specialize these equations for the air-water case, (q=0), although in extending the analysis to the oil-water system we will require the full equations. Thus, (with q=0):

$$
\left\{ -p^{\ell} + \frac{(1-\xi_x^2)}{(1+\xi_x^2)} \varepsilon^2 e_{yy} - 2 \frac{\xi_x}{(1+\xi_x^2)} \varepsilon^2 e_{xy} = \frac{\xi_{xx}\omega\sigma}{(1+\xi_x^2)^{3/2}} \right\} \quad (8.7a)
$$
$$
y=\xi
$$

$$
\left\{ \frac{(1-\xi_x^2)}{(1+\xi_x^2)} \varepsilon^2 e_{xy} + 2 \frac{\xi_x}{(1+\xi_x^2)} \varepsilon^2 e_{yy} = \zeta\xi_s \right\} \quad (8.8a)
$$
$$
y=\xi
$$

Or, carrying the expansion through $O(\alpha^2)$:

$$\{-p^{\ell} + \varepsilon^2 e_{yy} - 2\varepsilon^2 \xi_x e_{xy} = \xi_{xx}\omega_\sigma + O(\alpha^3)\}\Big|_{y=\xi} \qquad (8.7b)$$

$$\{\varepsilon^2 e_{xy} + 2\varepsilon^2 \xi_x e_{yy} = \zeta\xi_s + O(\alpha^3)\}\Big|_{y=\xi} \qquad (8.8b)$$

We may also recast the viscous stress term in Equation (8.4) using the rate of strain components with the result:

$$\varepsilon^2 \nabla^2 w \equiv \varepsilon^2 \frac{\partial}{\partial x} e_{yx} + \varepsilon^2 \frac{\partial}{\partial y} e_{yy} \qquad (8.9)$$

so,

$$-\varepsilon^2 \int_{-y^*}^{\xi} \nabla^2 w \, dy = -\varepsilon^2 e_{yy}\Big|_{-y^*}^{\xi} - \varepsilon^2 \int_{-y^*}^{\xi} \frac{\partial}{\partial x} e_{yx} \, dy \qquad (8.10)$$

Thus, we may write Equation (8.4) as:

$$p(-y^*) = -\xi_{xx}\omega_\sigma - 2\varepsilon^2 \xi_x e_{xy} + \varepsilon^2 e_{yy}\Big| + \omega_g(\xi + y^*)$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial t^2}(\xi^2) - \{y\frac{\partial w}{\partial t} + w^2\}\Big|_{y=-y^*} + \int_{-y^*}^{0}\{\frac{\partial}{\partial x}[y\frac{\partial u}{\partial t} + uw]\}dy - \varepsilon^2 e_{yy}\Big|_{-y^*}^{\xi}$$

$$- \varepsilon^2 \int_{-y^*}^{\xi} \frac{\partial}{\partial x} e_{yx} dy + O(\alpha^3) \qquad (8.11)$$

where the upper limit on the first integral on the right hand side has been replaced with y=0 on account of the $O(\alpha^2)$ magnitude of the integrand in this region.

We can in like fashion integrate the horizontal momentum equation, Equation (8.1), from -y* to ξ. We will subsequently specialize our choice of -y* by requiring that it be sufficiently far from the surface so that the first order rotational boundary layer flow will be negligible. Thus, the following calculations are aimed at determining the flow properties at the outer edge of the Stokes boundary layer. By invoking Liebnitz's theorem we can take the partial derivatives out of the integral and in the process generate a number of terms whose values are known from the boundary conditions at y=ξ. For example, the left hand side of Equation (8.1) becomes:

$$\int_{-y^*}^{\xi} \{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial y} \} \, dy = \frac{\partial}{\partial t} \int_{-y^*}^{\xi} u \, dy + \frac{\partial}{\partial x} \int_{-y^*}^{\xi} u^2 dy$$

$$- \{uw - u\dot{\xi} - u^2 \xi_x \} - uw \Big|_{y=\xi \qquad y=-y^*} \qquad (8.12)$$

while the pressure term on the right hand side provides us
with:

$$- \int_{-y^*}^{\xi} \frac{\partial p}{\partial x} \, dy = - \frac{\partial}{\partial x} \int_{-y^*}^{\xi} p \, dy + \{p\xi_x\}_{y=\xi} \qquad (8.13)$$

The bracketed term on the right hand side of
Equation (8.12) can be shown to be identically zero from
the surface kinematic condition, Equation (8.5). Thus the
integrated equation can be put in the following form:

$$\frac{\partial}{\partial t} \int_{-y^*}^{\xi} u \, dy + \frac{\partial}{\partial x} \int_{-y^*}^{\xi} (u^2 + p) \, dy$$

$$= [uw]_{y=-y^*} + [p\xi_x]_{y=\xi} + \epsilon^2 \int_{-y^*}^{\xi} \nabla^2 u \, dy \qquad (8.14)$$

This equation is exact. We may again rewrite
the viscous stress term using the rate of strain tensor
components. Thus, Equation (8.14) may be written:

$$\frac{\partial}{\partial t} \int_{-y^*}^{\xi} u \, dy + \frac{\partial}{\partial x} \int_{-y^*}^{\xi} u^2 \, dy = [uw]_{y=-y^*} + \xi_x(p - \epsilon^2 e_{xx})_{y=\xi} + \epsilon^2 e_{xy} \Big|_{-y^*}^{\xi}$$

$$+ \frac{\partial}{\partial x} \int_{-y*}^{\xi} (-p + \varepsilon^2 e_{yx}) dy \qquad (8.15)$$

or, rearranging a few terms:

$$\frac{\partial}{\partial t} \int_{-y*}^{\xi} u\,dy + \frac{\partial}{\partial x} \int_{-y*}^{\xi} u^2 dy = [uw]\Big|_{y=-y*} + \varepsilon^2 e_{xy}\Big|_{-y*}^{\xi}$$

$$+ \int_{-y*}^{\xi} \frac{\partial}{\partial x} (-p + \varepsilon^2 e_{xy}) dy \qquad (8.16)$$

Equation (8.16) states quite simply that the time rate of
change of momentum in the vertical plane plus the local
divergence of the momentum convected across the plane
equals the rate at which horizontal momentum is convected
up into the surface of the plane by the vertical velocity
at the lower edge, plus the stresses applied in the X
direction at the upper and lower edges plus the gradient
of the normal stress integrated over the face of the plane.
Our subsequent discussion will order these terms for the
monochromatic wave problem, and it will be useful to
remember this interpretation of the physical roles played

by the different terms.

Before getting into these arguments, it is
appropriate to digress at this point to comment on a
feature of one of the Longuet-Higgins and Stewart papers
(1964, pages 540-541) that might appear to be at variance
with the present analysis and the equations above.
Specifically, these authors briefly sketched some rough
arguments to justify the inclusion of an impulsive pres-
sure term at the upper surface representing the surface
tension contribution to the horizontal momentum transport.
After giving the matter some thought, it appears to me
that this technique was adopted by Longuet-Higgins and
Stewart in order to accommodate their previous development
of the pressure term in which they did not include the
surface tension explicitly in the normal stress condition.
Since we use the surface tension term explicitly, we need
not conjure up arguments at this point to account for its
absence. Thus the terms discussed by Longuet-Higgins and
Stewart are not to be found in the development.

The integral of the last term in Equation (8.16),
$(-p + \varepsilon^2 e_{xy})$, has the following expansion in α:

$$-p + \varepsilon^2 e_{xx} = \alpha \sigma_1 + \alpha^2 \sigma_2 + \ldots \qquad (8.17)$$

where:

$$\alpha\sigma_1 = \omega_\sigma\xi_{xx} - \omega_g(\xi + y^*) - \epsilon^2 e_{yy}(-y^*) + \epsilon^2 e_{xx}(-y^*)$$

$$+ y^* \frac{\partial w}{\partial t} - \epsilon^2 \int_{-y^*}^{\xi} \frac{\partial}{\partial x} e_{yx} dy \qquad (8.18)$$

The last two terms in y^* are included above in part because we have not yet restricted the magnitude of y^*. Clearly, if $y^* = O(\alpha)$ then these terms will belong with $\alpha^2\sigma_2$.

Using this expansion and passing a one period time average over Equation (8.16), we have:

$$\frac{\partial}{\partial x} \int_{-y^*}^{0} \overline{u^2} dy = [\overline{uw}] + \epsilon^2 \overline{e}_{xy} \Big|_{-y^*}^{\xi} + \frac{\partial}{\partial x} \int_{-y^*}^{0} \alpha^2 \overline{\sigma}_2 dy$$

$$+ \int_{-y^*}^{\xi} \overline{\frac{\partial}{\partial x} \alpha\sigma_1} dy + O(\alpha^3) \qquad (8.19)$$

Now expand the last term by breaking the integration into two intervals:

$$\int_{-y^*}^{\overline{\xi}} \frac{\partial}{\partial x} \alpha\sigma_1 dy = \int_{-y^*}^{-y^o} \alpha \frac{\partial}{\partial x} \overline{\sigma_1} dy + \int_{-y_o}^{\overline{\xi}} \alpha \frac{\partial\sigma_1}{\partial x} dy$$

$$= \alpha \int_{-y_o}^{\overline{\xi}} \frac{\partial\sigma_1}{\partial x} dy \qquad (8.20)$$

where $-y_o$ is a level selected so that it is independent of x and t, and just far enough beneath the surface so that it is not uncovered by the wave (thus insuring $|y^o| = O(\alpha_y)$; and $\overline{\sigma_1} = 0$ on account of Equation (8.18).

If we assume irrotational, inviscid flow, the evaluation of the integral in Equation (8.20) would be rather straightforward. It corresponds, in fact, to S_{xx} [3] of equation (3) of section 2 of Longuet-Higgins and Stewart (1964). However, the presence of the viscous stress terms in σ_1, coupled with both the oscillatory behavior of the upper boundary and the singular nature of the boundary layer make it difficult to proceed for the case $\alpha \gg \varepsilon$. One possible approach would be to decompose σ_1 into irrotational and rotational components, evaluate the irrotational component explicitly with a Taylor's series about y=0, and make order of magnitude arguments

regarding the rotational contribution. Unfortunately, we do not have the requisite $O(\alpha)$ solution for the rotational flow when $\alpha \gg \varepsilon$, and so this technique would require further first order analysis to carry the argument through in a rigorous fashion.

Therefore, we will restrict the problem to the case $\alpha \ll \varepsilon$. This allows us to expand the integrand in terms of an inner boundary layer variable expansion, where the magnitudes of the various terms are known from our previous discussion of the first order problem. In this case we have:

$$\alpha \, \frac{\partial \sigma_1}{\partial x} = \alpha \, \frac{\partial \sigma_1}{\partial x} \bigg|_{y=0} + \frac{\alpha}{\varepsilon} \, \frac{\partial^2 \sigma_1}{\partial x \partial y} \bigg|_{y=0} y + \ldots \qquad (8.21)$$

We may evaluate the expansion coefficients as follows:

$$\alpha \, \frac{\partial \alpha_1}{\partial x} \bigg|_{y=0} = \omega_o \xi_{xxx} - \omega_g \xi_x + 2\varepsilon^2 \, \frac{\partial}{\partial x} \, e_{xx}(0) \qquad (8.22)$$

$$\frac{\alpha}{\varepsilon} \, \frac{\partial^2 \sigma_1}{\partial x \partial y} \bigg|_{y=0} = \frac{1}{\varepsilon} \left\{ 2\varepsilon^2 \frac{\partial^2}{\partial x \partial y} \, e_{xx}(0) + \varepsilon \frac{\partial^2 w(0)}{\partial x \partial t} - \varepsilon^2 \frac{\partial^2}{\partial x^2} e_{xy} \right\} \qquad (8.23)$$

where we have used Equation (8.18) to evaluate $\alpha\sigma_1$.

Thus, to $O(\alpha^2)$, we have:

$$\alpha \int_{-y_o}^{\overline{\xi}} \frac{\partial}{\partial x} \sigma_1 dy = \overline{[\omega_\sigma \xi_{xxx} - \omega_g \xi_x + 2\varepsilon^2 \frac{\partial}{\partial x} e_{xx}(0)]\xi} \qquad (8.24)$$

and Equation (8.18) becomes:

$$\int_{-y^*}^{0} \frac{\partial}{\partial x} (\overline{u}^2 - \alpha^2 \overline{\sigma}_2) dy = [\overline{uw}] \Big|_{-y^*} + \varepsilon^2 \overline{e}_{xy} \Big|_{-y^*}^{\xi} + \omega_\sigma \overline{\xi_{xxx}\xi}$$

$$- \omega_g \overline{\xi_x \xi} + 2\varepsilon^2 \frac{\partial}{\partial x} \overline{e_{xx}(0)\xi} \qquad (8.25)$$

This equation may be viewed as specifying the value of $\overline{e}_{xy}$ at $y=-y^*$ since many of the other terms are known explicitly as boundary conditions or as results of the first order analysis. The only exception to this rule being the integral on the left hand side which we shall show to be small compared to the rest provided the wave is being strongly damped, and provided $-y^*$ is chosen so as to be in the vicinity of the lower edge of the Stokes layer.

Physically, the equation will tell us that in the absence of second order rotational flows not hypothesized

at this point, the gradient of the first order, x directed

normal stress, $[-p + \varepsilon^2 e_{xx}$ in Equation (8.16)], imparts

a net second order momentum to the fluid lying in the

region of the wave crests. Momentum must be drawn from

the region as rapidly as it is created, and the mechanism

established for this purpose is the steady tangential

stress at y=-y*. Since the horizontal scales may be

presumed to be large compared to the vertical scales, a

reasonable approximation to this steady stress is:

$$\varepsilon^2 \bar{e}_{xy}\bigg|_{y=-y^*} \simeq \varepsilon^2 \left.\frac{\partial \bar{u}}{\partial y}\right|_{y=-y^*} \tag{8.26}$$

To show the relative magnitudes of the terms in

equation (8.25), assume that the free surface elevation

and wave number are given by:

$$\xi = R\ell\{\alpha\xi_o e^{iKx-it}\}, \text{ and} \tag{8.27}$$

$$K = 1 + \varepsilon K_1 + \varepsilon^2 K_2 + \dots \tag{8.28}$$

where K_1 is non-zero only for the monolayer or oil layer

case.

Then the terms $\omega_\sigma \overline{\xi_{xxx}\zeta}$ and $- \omega_g \overline{\zeta_x\xi}$ may be evaluated as follows:

$$\omega_\sigma\overline{\xi_{xxx}\xi} = \omega_\sigma \frac{1}{2} R\ell\{(-iK^3)\xi_o e^{iKx}\xi_o^{*}e^{-iK^*x}\alpha^2\} =$$

$$\frac{1}{2}\alpha^2|\xi_o|^2\omega_\sigma 3\varepsilon \; Im\{K_1\}e^{i(K-K^*)x} \tag{8.29}$$

$$- \omega_g\overline{\xi_x\xi} = - \omega_g \frac{1}{2} R\ell\{(iK)\xi_o e^{iKx}\xi_o^{*}e^{-iK^*x}\} =$$

$$\frac{1}{2}\alpha^2|\xi_o|^2\omega_g\varepsilon \; Im\{K_1\}e^{i(K-K^*)x} \tag{8.30}$$

where ξ_o is an arbitrary factor $O(1)$ of no concern; the superscript * indicates complex conjugates; and

$$i(K-K^*) = - 2\varepsilon \; Im\{K_1\} \equiv - \varepsilon\beta. \tag{8.31}$$

Thus the near-surface pressure gradient terms can contribute a term $O(\alpha^2\varepsilon)$ to the right hand side of equation (8.25) when the waves are strongly damped, as in the monolayer case.

The $\overline{uw}$ term in equation (8.25) is easily evaluated by noting that in the irrotational core region (i.e., at and below $y = -y^*$), the $O(\alpha)$ wave velocity components will be exactly 90 degrees out of phase since they spring from the x and y derivatives of exp. $\{iKx + Ky\}$. Thus their time average at the depth $-y^*$ will be identically zero.

The remaining terms require a more detailed examination. The integrand of the left hand term of equation (8.25) may be manipulated as follows:

$$\frac{\partial}{\partial x}(\overline{u}^2 - \alpha^2\overline{\sigma}_2) = \frac{\partial}{\partial x}\left\{\overline{u}^2 - \overline{w}^2 - 2\varepsilon^2\overline{\xi_x e_{xy}}\Big|_{y=\xi} - 2\varepsilon^2\overline{e}_{yy} + \int_{-y^*}^{0}\frac{\partial}{\partial x}\overline{uw}\,dy - \varepsilon^2\int_{-y^*}^{\xi}\frac{\partial}{\partial x}e_{yx}\,dy\right\} =$$

$$\frac{\partial}{\partial x}\left\{\overline{u}^2 - \overline{w}^2 - 2\varepsilon^2\overline{e}_{yy} - \varepsilon^2\overline{\xi_x e_{xy}}\Big|_{y=\xi} + \int_{-y^*}\frac{\partial}{\partial x}\overline{uw}\,dy - \varepsilon^2\frac{\partial}{\partial x}\int_{-y^*}^{\xi}e_{yx}dy\right\} \qquad (8.32)$$

Consider first the terms $\partial/\partial x\ (\overline{u}^2 - \overline{w}^2)$ and $-\int_{-y^*}^{o} \partial/\partial x\,(\overline{uw})\,dy$.
Recall that we are here considering only wave related
phenomena. Thus u and w are assumed to be directly
associated with the first order wave velocities. In this
case, only in a region of $O(\epsilon)$ will $\overline{u}^2$ differ significantly
from $\overline{w}^2$, and only in this region will $\overline{uw}$ be nonzero.
Further, these quantities will depend upon x much like
the surface pressure terms, (i.e., like $e^{-\beta\epsilon x}$, see
equation (8.29)). So taking the 'x' derivative of these
terms will introduce additional factors of $O(\epsilon)$. This
implies that:

$$\int_{-y^*}^{o} \frac{\partial}{\partial x}\ \overline{u}^2 - \overline{w}^2 \ dy = O(\alpha^2\epsilon^2) \tag{8.33}$$

$$\int_{-y^*}^{o} \int_{-y'}^{o} \frac{\partial^2}{\partial x^3}\ \overline{uw}\ dy\ dy' = O(\alpha^2\epsilon^3 y^*) \tag{8.34}$$

We have assumed that the wave amplitude is small compared
to ϵ $(\alpha \ll \epsilon)$, so y^* may be chosen to be $O(\epsilon)$. Thus
the second term will be negligible compared to the first,
and both will be small compared to the surface pressure
term.

The remaining terms in the integrand, equation (8.32), are:

$$\frac{\partial}{\partial x}\left\{ -2\varepsilon^2\overline{e}_{yy} + \varepsilon^2 \left.\overline{\xi_x e_{xy}}\right|_{y=\xi} - \varepsilon^2 \frac{\partial}{\partial x}\int_{-y^*}^{\xi} e_{xy}dy \right\}$$

These terms are more difficult to order as they are intimately related to the properties of the second order flow. In particular, if $\partial^2\overline{u}/\partial x^2$ is large there would be a substantial transport of horizontal momentum by viscous stresses, and our ordering techniques collapse. It seems unlikely, however, that this could ever be the case physically, for it would suggest vigorous motions within the spill region, and there is little evidence that would support such an idea (except perhaps in isolated regions near a leading or trailing edge). In fact, if our observations regarding the constancy of the area of the thick region of an oil spill suggest anything, they suggest that $\partial\overline{u}/\partial x$ and $\partial^2\overline{u}/\partial x^2$ are negligible over most of the region. Thus we shall make the assumption that $\partial\overline{u}/\partial x$ is negligible (say $O(\alpha^2)$ or smaller) in the near surface region (i.e. from $y = \xi$ to $y = -y^*$). This doesn't imply that $\partial\overline{u}/\partial x$

is negligible everywhere, however, because terms like $\partial \bar{u}/\partial x$ are essential in establishing the secondary flow beneath the Stokes layer.

With this assumption regarding $\partial \bar{u}/\partial x$ we also assume $\partial \bar{w}/\partial y$ is small. This allows us to determine that $\bar{w}$ and its 'x' derivatives are small everywhere in the Stokes layer because at $y = 0$, we have the Taylor's expansion of the surface kinematic condition, equation (8.5), to tell us that

$$\bar{w}\Big|_{y=0} = \frac{\partial}{\partial x} (\overline{u\xi}) = O(\alpha^2 \epsilon). \tag{8.35}$$

Since $\partial \bar{w}/\partial y$ is small, $\bar{w}$ at a point $O(\epsilon)$ beneath the surface is only little changed from $\bar{w}$ at $y = 0$, and so it too is small. With these assumptions we find:

$$\int_{-y*}^{O} - \frac{\partial}{\partial x} 2\epsilon^2 \bar{e}_{yy} \, dy = O(\alpha^2 \epsilon^2 y*) \tag{8.36}$$

$$\int_{-y*}^{O} \frac{\partial}{\partial x} \epsilon^2 \overline{\xi_x e_{xy}}\Big|_{=y} \, dy = O(\alpha^2 \epsilon^2 y*) \tag{8.37}$$

$$\int\limits_{-y^*}^{0} - \, \epsilon^2 \, \frac{\partial^2}{\partial x^2} \int\limits_{-y'}^{\xi} e_{xy} \, dy \; =$$

$$\int\limits_{-y^*}^{0} - \, \epsilon \, \frac{\partial}{\partial x} \left[\overline{u}(\xi) \, - \, \overline{u}(y') \, - \, \xi \, \overline{\frac{\partial w}{\partial x}} \, - \right.$$

$$\left. \int\limits_{-y'}^{0} \frac{\partial \overline{w}}{\partial x} \, dy \right] \, dy' \; = \; O(\alpha^2 \epsilon^4 y^*) \qquad (8.38)$$

Thus we have finally:

$$\epsilon^2 \, \frac{\partial \overline{u}}{\partial y} \Bigg|_{y=-y^*} \; = \; \epsilon^2 \overline{e}_{xy} \Bigg|_{y=\xi} \; +$$

$$\frac{1}{2} \, \alpha^2 \, \epsilon \, \text{Im} \, \{K_1\} \, \overline{e}^{\beta \epsilon x} \left[\omega_g + 3\omega_\sigma \right] \; +$$

$$O(\alpha^2 \epsilon^2) \qquad (8.39)$$

where we have used equations (8.29) - (8.30) in the last term on the right.

Equation (8.8b) provides us with the time averaged rate of strain term at the surface, whence:

$$\left\{ \varepsilon^2\overline{e_{xy}} = \overline{\zeta \xi_s} - 2\varepsilon^2\overline{\xi_x e_{yy}} \right\}_{y=\xi} \tag{8.40}$$

The last term in the bracket is $O(\alpha^2\varepsilon^2)$ so equation (8.39) becomes:

$$\varepsilon^2 \left. \frac{\partial \overline{u}}{\partial y} \right|_{y=-y^*} = \overline{\zeta \xi_s} +$$

$$\tag{8.39a}$$

$$\frac{1}{2} \alpha^2 \varepsilon \text{ Im } \{K_1\}\overline{e}^{-\beta\varepsilon x} \left[\omega_g + 3\omega_\sigma \right] + O(\alpha^2\varepsilon^2)$$

This equation has the rather novel parameter $\overline{\zeta \xi_s}$, which is the stress applied to the surface by a spatially varying steady accumulation of surfactants on the surface. In the crude model we have adopted here, there is no good way of establishing what this parameter is because we have neglected diffusion and extraction of the surfactants. In fact, in the simple strain model used in Section 6, we find that the surfactant concentration either oscillates sinusoidally or grows linearly with time.

In an actual oil spill, we would expect regions of higher "pressure", i.e. higher surface

concentrations of surfactants, to be regions where dillusion into the two adjoining bulk phases is greatly enhanced. Thus, localized accumulations of surfactants caused by convergence of the surface velocity would be relaxed by the diffusional transfer. Since the time scale would be given by $(\partial \bar{u}/\partial x)^{-1}$, where u and x are dimensional, and since $\partial \bar{u}/\partial x$ is presumed to be small, the relaxation effect could be sufficient to make this contribution negligible. But this term is more complicated than just this, so we shall come back to it later in the next section.

We may extend the analysis to the oil-water system rather simply. Utilizing the same integration techniques, we have the following equation for the pressure within the oil layer (i.e. $\xi_\ell < y < D + \xi_u$):

$$p_u(y) = p_u(D + \xi_u) + \omega_g(D + \xi_u - y) +$$

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\left[\xi_u^2\right] - \left.\left[y\frac{\partial w}{\partial t} + w^2\right]\right|_y +$$

$$\int_y^{D+\xi_u} \frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial t} + uw\right] dy - \left. \varepsilon^2 e_{yy}^u \right|_y^{D+\xi_u} -$$

$$\varepsilon^2 \int_y^{D+\xi_u} \frac{\partial}{\partial x} e_{yx}^u \, dy \qquad (8.41)$$

where ξ_ℓ defines the oil-water interface;

$D + \xi_u$ defines the air-oil interface;

ε^2 is now calculated using the oil layer's density and viscosity; and

$$\left\{ y\, \frac{\partial w}{\partial t} - w^2 \right\}_{y=\xi_\ell} = \frac{1}{2}\, \frac{\partial^2}{\partial t^3}\, (\xi_\ell^2) + O(\alpha^3).$$

Likewise, the horizontal momentum equation may be integrated over the oil layer to yield:

$$\frac{\partial}{\partial t} \int_{\xi_\ell}^{D+\xi_u} u\, dy + \frac{\partial}{\partial x} \int_{\xi_\ell}^{D+\xi_u} u^2 dy = \varepsilon^2 e_{xy}^u \Big|_{\xi_\ell}^{D+\xi_u} +$$

$$\int_{\xi_\ell}^{D+\xi_u} \frac{\partial}{\partial x} (-p_u + \varepsilon^2 d_{yy}^u)\, dy \qquad (8.42)$$

As in the analysis above, we again find that the time averaged tangential stress term is balanced by the time averaged integrals of the first order pressure gradient terms in the vicinity of the oscillating interface.

$$0 = \overline{e_{xy}^u}\Big|_{\xi_\ell}^{D+\xi_u} - \overline{\frac{\partial p_u}{\partial x}(D)\xi_u} + \overline{\frac{\partial p_u}{\partial x}}\Big|_{(0)} \xi_\ell +$$

$$O(\alpha^2 \varepsilon^2) \qquad (8.43)$$

The sign is reversed for the term springing from the integration at the lower boundary because the integration is carried out from the interface upwards. Thus

$$\varepsilon^2 \overline{e}^u_{xy}\bigg|_{\xi_\ell} = \varepsilon^2 \overline{e}^u_{xy}\bigg|_{D+\xi_u} + \overline{\xi_{u_{xxx}} \xi_u} \frac{\omega_\sigma}{q} +$$

$$\omega_g \overline{\xi_{u_x} \zeta_u} - \left(\frac{\omega^u_\sigma}{q} \overline{\xi_{u_{xxx}} \xi_\ell} - \omega_g \overline{\xi_{u_x} \xi_\ell} \right) \qquad (8.44)$$

where

$$P_u(D) = -\frac{\omega_\sigma}{q} \xi_{xx} + \omega_g(D + \xi_u) + O(\alpha\varepsilon^2) =$$

$$p(o) + O(\alpha\varepsilon^2); \text{ and}$$

where the term ω^u_σ is defined using the lower fluid density, thus the requirement for the q in the denominator.

In Section 7 we pointed out that the upper and lower interface oscillated nearly in unison, the first non-zero difference being of order ε^u. Thus $\xi_u - \xi_\ell$ is small and;

$$\varepsilon^2 \overline{e}^u_{xy}\bigg|_{\xi_\ell} \simeq \varepsilon^2 \overline{e}^u_{xy}\bigg|_{D+\xi_u} + O(\alpha^2\varepsilon^2) \qquad (8.45)$$

We may now use equation (8.41) with y taken as ξ_ℓ to determine the value of p on the upper side of the interface. This we may then use in the interfacial normal

stress condition, equation (8.7), to establish the $O(\alpha)$ pressure to be used in equation (8.22). Thus,

$$\alpha \left.\frac{\partial \sigma_1}{\partial_x}\right|_{y=0} = \xi_{\ell_{xxx}} \omega_\sigma^i - \omega_g \xi_{\ell_x} -$$

$$q \left\{ -\frac{\omega_\sigma^u}{q} \zeta_{u_{xxx}} + \omega_g (\xi_{u_x} - \xi_{\ell_x}) \right\} +$$

$$O(\alpha \epsilon^2) \qquad\qquad (8.46)$$

The tangential stress condition to be applied at the oil water interface is obtained from (8.8), whence:

$$\epsilon^2 \overline{e_{xy}^\ell}\bigg|_{y=\xi_\ell} = \overline{\zeta e_s} + q\epsilon^{u_2} \overline{e_{xy}^u}\bigg|_{y=\xi_\ell} =$$

$$\overline{\zeta e_s} + q\epsilon^{u_2} \overline{e_{xy}^u}\bigg|_{y=D+\xi_u} \qquad\qquad (8.47)$$

The oil-air interface is assumed to be non-elastic, thus:

$$\epsilon^{u_2} \overline{e_{xy}^u}\bigg|_{y=D+\xi_u} = O(\alpha^2 \epsilon^2) \qquad\qquad (8.48)$$

We may now insert the above results in equation (8.25),
through out the terms we know to be small, and

$$\varepsilon^2 \frac{\partial \overline{u}}{\partial y}\bigg|_{y=y^*} = \overline{\zeta\xi}_s + \frac{1}{2}\alpha^2\varepsilon \ \ell m\{K_1\}\overline{e}^{\beta\varepsilon x}\{\omega_g + 3(\omega_\sigma^i + \omega_\sigma^u)\}$$

$$+ \ O(\alpha^2\varepsilon^2) \tag{8.49}$$

which should be compared with (8.39a). Thus, the effect
of this oil layer is negligible under the assumptions
of this analysis, which are principally that $\partial\overline{u}/\partial x$ be
small, $O(\alpha^2)$ say, and the $\alpha \ll \varepsilon$.

It is also possible to derive the same results
from an assumptotic power series expansion for the
time steady second order stream function. For the air
water case (i.e., the monolayer problem) the analysis
proceeds from an expansion of the form:

$$\Psi = R\ell\{\alpha e^{iKx-it}\left[\psi_{10} + \varepsilon\psi_{11}+..\right]\} +$$

$$\alpha^2\left[\psi_{20} + \varepsilon\psi_{21}+..\right]\overline{e}^{\beta\varepsilon x} + \alpha^2\Psi_2(x,y,t) \tag{8.50}$$

where the subscripted indices now indicate the order of
α and ε (whereas in previous sections they indicated the
order of ε^u and ε^ℓ);

$$\psi_{10} = a_{10};$$

(8.51)

$$\psi_{11} = a_{11} + b_{11}y + C_{11}e^{-(1-i)y/\sqrt{2}} \quad ; \text{ and}$$

(8.52)

$$\psi_{12} = a_{12} + b_{12}y + C_{12}e^{-(1-i)y/\sqrt{2}} +$$

$$\frac{1}{2} a_{10}y^2 .$$

(8.53)

Here the forms of equation (8.51) through (8.53) are determined from Section 6. Inserting the assumed expansions into the vorticity equation (4.9), we find the following governing equations for the first two terms in the steady second order expansion:

$$\overline{\psi}_{20}^{IV} = \frac{1}{2} R \left\{ ia_{11}^{*} \frac{(1+i)}{\sqrt{2}} C_{11} e^{\frac{(1-i)}{\sqrt{2}} y} \right\}$$

(8.54)

$$\overline{\psi}_{21}^{IV} = -\frac{1}{2} e^{-\beta\varepsilon x} \left\{ \sqrt{2}|C_{11}|^2 \overline{e}^{\sqrt{2}y} + \right.$$

$$R\ell \left[a_{10}^{*} (C_{11} + \frac{(1-i)}{\sqrt{2}} C_{12})e^{-(1-i)y/\sqrt{2}} - \right.$$

$$\left. C_{11}^{*} \frac{(1+i)}{\sqrt{2}} (C_{11} + a_{10}y)e^{-(1+i)y/\sqrt{2}} \right] \right\}$$

(8.55)

The boundary conditions are obtained from equations (8.5), (8.7b) and (8.8b). The functions $\bar{\psi}_{21}$ and $\bar{\psi}_{20}$ are obtained by simply integrating (8.54) and (8.55) subject to the boundary condition. Prior to incorporating the boundary condition, ψ_{20} is found to be:

$$\bar{\psi}_{20} = \frac{1}{2} R\ell \left\{ a_{10}^* \frac{(1-i)}{\sqrt{2}} c_{11} e^{-(1-i)y/\sqrt{2}} \right\} +$$

$$\frac{1}{6} a_{20} y^3 + \frac{1}{2} b_{20} y^2 + c_{20} y + d_{20} \qquad (8.56)$$

For our discussion the critical integration constant is b_{20}, because this constant determines the value of $\partial\bar{u}/\partial y$ at the outer edge of the Stokes region. This constant is determined from equation (8.8b) which provides the following condition on $\psi_{20_{yy}}$ at $y = 0$.

$$\varepsilon^2 \left[\frac{\partial^2 u}{\partial y^2} \xi + \frac{1}{\varepsilon^2} \underset{20}{\overset{II}{\psi}}(0) \bar{e}^{\beta \varepsilon x} + 4\overline{\xi_x} \frac{\partial w}{\partial y} \right] = \overline{\zeta \xi_s} \qquad (8.57)$$

We argued in Section 6 that ζ/ε was typically $O(1)$. Thus we might reasonably consider the right hand side of (8.57), $\overline{\zeta \xi_s}$, as being $O(\alpha^2)$, and in this case we have:

$$\frac{1}{2} \, R\ell \left\{ \xi^* \, \frac{(1-i)}{\sqrt{2}} \, c_{11} e^{\frac{(1-i)}{\sqrt{2}}y} \right\} +$$

$$\text{(8.58)}$$

$$\frac{1}{2} \, R\ell \left\{ a_{10}^* \, \frac{(1+i)}{\sqrt{2}} \, c_{11} e^{-\frac{(1+i)}{\sqrt{2}}y} \right\} + b_{20} = 0$$

Now the surface kinematic condition can be used to establish the identity $\xi = - a_{10}$, being careful to remember that the inner variable is positive downwards (so $w = + \psi_x$). Thus the bracketed terms cancel and we are left with

$$b_{20} = 0 \qquad\qquad (8.59)$$

Thus the lowest order steady flow does not contribute to the shear stress at the outer edge of the Stokes layer.

Similar calculations have also been carried out for equation (8.55). However, the equations become rather long and involved, and the previous discussion leading to (9.39a) serves just as well to explain the physical mechanisms involved. Moreover, the answer is the same, and much more easily obtained in the previous formulation.

Since the mathematical arguments have been rather complicated in this section, it is useful here to summarize the results and comment on their possible importance. Of greatest importance is the observation that the assymptotic behavior of the steady Stoke's boundary layer flow is characterized by the formation of a steady velocity shear. This shear is related at lowest order to the $O(\epsilon)$ decay properties of the free surface. We assumed an exponentially decaying wave in equation (8.27) and this leads to the exponential form exhibited in equations (8.39) and (8.39a). It is possible to envisage other forms for the free surface profile that would lead to slightly different 'x' dependency for the velocity shear. We mentioned that we did not have a very good model of the (possible) steady shear stress induced by the compaction of the surfactants on the interface. Thus the term $\overline{\zeta \xi_s}$ in (8.39a) is mostly to remind us that the problem may be more complicated than we have assumed up to this point. We have shown that the oil layer and the monolayer will behave similarly, the only difference being the use of the terms $\omega_\sigma^i + \omega_\sigma^u$ in place of ω_σ (as we could have anticipated from Section 7).

The restrictions on the analysis are of two

forms. One requires $\alpha \ll \epsilon$, and this is indeed a severe

constraint with respect to application of the results to

the open sea. This assumption was invoked at equation

(8.20) and a useful next step would be to generalize

the arguments of equations (8.21) to (8.24) to accomodate

$\alpha \sim \epsilon$ and $\alpha \gg \epsilon$. This may not be as difficult as it

first seems, although it would seem unwise to simply

assume that the result will follow.

The other restriction was that $\partial \bar{u}/\partial x$ be small

within the Stoke's boundary layer region. This might

serve as a boundary condition on the outer flow and so be

satisifed automatically. However, it would be most

helpful to have more detailed observations from field

experiments in order to get some idea of the possible

convergence or divergence of the flow field at the

interface. Radioactivity labeled surfactants would seem

to have considerable utility in this regard, although

there may well be environmental problems associated with

the release of such compounds that I have not considered.

9. APPLICATIONS, RECOMMENDATIONS FOR FURTHER STUDY AND CONCLUSIONS

Our principal concern in the sections above has been to provide the reader with a reasonably comprehensive discussion of those features of an oil spill that appear to be of importance in the first order oil-wave interactions. We have attempted to document the various roles played by viscosity, frequency and the interfacial elasticity. We have also examined the characteristics of the second order steady flow within the Stokes boundary layer. We have not attempted to carry the problem beyond this point because of the multitude of uncertainities that beset our knowledge of the important physical properties of the oil-water system.

One of the key points that remains to be analyzed and one of the reasons for undertaking this work in the first place is what is the nature of wave-induced transport of the oil? It seems quite reasonable to expect that this mass transport will be determined by the secondary boundary layer that would be formed beneath the Stokes region. Stuart (66), for example, speculated on the nature of such a secondary surface boundary layer for ordinary water waves.

In our problem, it would seem that the near

surface boundary conditions to be applied to such a
secondary flow would involve either a horizontal
velocity or a horizontal velocity gradient, and a
horizontal velocity shear whose magnitude could be
determined from a knowledge of the local wave amplitude
and decay length (e.g., see equation 8.39). Unfortunately,
our discussion regarding the sensitivity of the decay
length to small perturbations in oil thickness and
monolayer elasticity would make it seem that in the
absence of further data on actual oil spill behavior
almost any sort of velocity shear might be hypothesized.
It must also be remembered that for the results of
Section 8 to apply, it is essential that the value of
$\frac{\partial \bar{u}}{\partial x}$ be kept reasonably small in the vicinity of the Stokes
boundary layer least the outer flow begin to interact
with the inner flow, thereby rendering the velocity
shear boundary condition of Section 8 inapplicable.

If through experiment or analysis the
character of these boundary conditions are determined,
then a most appropriate mathematical technique for
calculating the secondary flow is the Wiener-Hopf
method as discussed by Carrier, et al (1966, pg. 376-382).

An important research priority would therefore
appear to be to establish some quantitative idea of

the steady flow properties of an oil spill. Is the spill
really composed of many small patches? Is there a
substantial convergence of oil flow at the interface
(i.e., is $\frac{\partial \bar{u}}{\partial x}$ large)? Does the thickness vary widely,
and are there preculiar thickness distributions at the
windward edge of the oil slick suggesting the presence of
the depth induced resonance discussed in Section 7?

Additionally, we require a much more comprehensive
understanding of the adsorption characteristics of
surfactants in real oils. What are the E_0's, β's and
δ's discussed in Section 2, and how do they vary with
time as the spill weathers?

Finally we require experimental verification
of the formulae of Section 7. This may require the
extension of the theory to include soluble surfactants,
as these might be the rule rather than the exception
for oil-water systems. Of course, the critical parameter
with respect to the solubility of a surfactant is the
product $\beta\delta$ of Section 2, and this can be made as small
as we wish by increasing the frequency. Thus, the
analysis of Section 7 can be applied directly provided
only that the frequency be high enough.

Prior to undertaking the analysis reported
on above, the author performed some simple calculations

equating the momentum lost by a decaying wave to the
momentum perturbation induced by a flat plate steadily
translating. This calculation revealed that the plate
velocity should equal the phase velocity of the wave
times the wave slope to the 4/3's power times a constant
O(1). It was anticipated that the analysis above would
provide a more substantial means of calculating this
velocity. As we have seen, the problem is actually much
to complicated to allow us to make the transition from
our present understanding of the wave-oil interaction
to the secondary boundary layer in the absence of
additional experimental information. The problem is an
interesting one, however, and it may lead to ideas that
might prove of use in many related fields. It is my
hope that some of these problems might be studied in
the near future, and a comprehensive model constructed
around the outlines of this paper.

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Appendix A: Matching and Boundary Condition Expansions for the Monolayer Problem

Stream Function Expansions (Equations 4.10 and 4.11)

$$\Psi^u = \mathrm{Re}\left\{\alpha\, e^{iKx - i\sigma t}\left[\psi^u_{00}(y) + \epsilon^1\psi^u_{01}(y) + \epsilon^u\psi^u_{10}(y) + \cdots\right]\right\} + \theta(\alpha^2).$$

$$\Psi^l = \mathrm{Re}\left\{\alpha\, e^{iKx - i\sigma t}\left[\psi^l_{00}(y) + \epsilon^u\psi^l_{01}(y) + \epsilon^1\psi^l_{10}(y) + \cdots\right]\right\} + \theta(\alpha^2).$$

Potential Function Expansions (Equations 4.15 and 4.16)

$$\Phi^u = \mathrm{Re}\left\{\alpha\, e^{iKx - i\sigma t}\left[A^u_{00} + \epsilon^1 A^u_{01} + \epsilon^u A^u_{10} + \cdots\right]e^{-Ky}\right\} + \theta(\alpha^2)$$

$$\Phi^l = \mathrm{Re}\left\{\alpha\, e^{iKx - i\sigma t}\left[A^l_{00} + \epsilon^u A^l_{01} + \epsilon^1 A^l_{10} + \cdots\right]e^{Ky}\right\} + \theta(\alpha^2)$$

where:

	Case 1	Case 2
K	$1 + \epsilon^1 K_{10} + \epsilon^u K_{01} + \cdots$	1
σ	1	$1 + \epsilon^1 \sigma_{10} + \epsilon^u \sigma_{01} + \cdots$

The functional forms of the ψ^{β}_{ij}'s in the stream function expansions are determined from 4.18 through 4.23b.

These funcions are:

Case 1

$$-\psi_{ij}^{\beta}(y) = a_{ij}^{\beta} + b_{ij}^{\beta}\,y + c_{ij}^{\beta}\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{ij}^{\beta}\,e^{\frac{(1-i)}{\sqrt{2}}y}$$

$$\beta = u, \ell \;; \quad i = 0, 1 \;; \quad j = 0, 1, 2, \cdots \;;$$

(Note, except when used as a subscript, i=SQRT(-1))

$$-\psi_{20}^{\beta}(y) = a_{20}^{\beta} + b_{20}^{\beta}\,y + c_{20}^{\beta}\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{20}^{\beta}\,e^{\frac{(1-i)}{\sqrt{2}}y} + \tfrac{1}{2}a_{00}^{\beta}\,y^{2}$$

$$+ \tfrac{1}{6}b_{00}^{\beta}\,y^{3} - c_{00}^{\beta}\frac{(1-i)}{2\sqrt{2}}\,y\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{00}^{\beta}\frac{(1+i)}{2\sqrt{2}}\,y\,e^{\frac{(1-i)}{\sqrt{2}}y}$$

Case 2

$$\psi_{00}^{\beta}(y) = a_{00}^{\beta} + b_{00}^{\beta}\,y + c_{00}^{\beta}\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{00}^{\beta}\,e^{\frac{(1-i)}{\sqrt{2}}y}$$

$$\psi_{01}^{u}(y) = a_{01}^{u} + b_{01}^{u}\,y + c_{01}^{u}\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{01}^{u}\,e^{\frac{(1-i)}{\sqrt{2}}y}$$

$$- c_{00}^{u}\,\sigma_{10}\frac{(1+i)}{2\sqrt{2}}\,y\,e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{00}^{u}\,\sigma_{10}\frac{(1+i)}{2\sqrt{2}}\,y\,e^{\frac{(1-i)}{\sqrt{2}}y}\ .$$

ETC.

The kinematic matching conditions require the velocity to vary continuously as we pass from the inner to the outer regions. These conditions are stated in

the text as equations 5.17 and 5.18. These equations are evaluated as follows.

5.17 (horizontal velocity)

$$\Phi^{\beta}_{x}\Big|_{y=\pm\epsilon^{\beta}y_{\infty}} = \frac{1}{\epsilon^{\beta}}\Psi^{\beta}_{y}\Big|_{y=y_{\infty}}$$

$$iK[A^{u}_{oo} + \epsilon^{2}A^{u}_{o1} + \epsilon^{u}A^{u}_{1o}\cdots] - iK[A^{u}_{oo} + \epsilon^{2}A^{u}_{o1}\cdots]K\epsilon^{u}y_{\infty} +$$

$$+ \frac{1}{2!}iK[A^{u}_{oo} + \epsilon^{2}A^{u}_{o1}\cdots]K^{2}(\epsilon^{u}y_{\infty})^{2} + \ldots$$

$$= \frac{1}{\epsilon^{u}}\Big[\Psi^{u'}_{oo}(y_{\infty}) + \epsilon^{2}\Psi^{u'}_{o1}(y_{\infty}) + \epsilon^{u}\Psi^{u'}_{1o}(y_{\infty}) + \ldots\Big]$$

Case 1:

$$iA^{u}_{oo} + \epsilon^{2}\{iA^{u}_{o1} + iK_{1o}A^{u}_{oo}\} + \epsilon^{u}\{iA^{u}_{1o} + iK_{o1}A^{u}_{oo} - iA^{u}_{oo}y_{\infty}\}\cdots$$

$$= \frac{1}{\epsilon^{u}}\Big[b^{u}_{oo} + \frac{(1-i)}{\sqrt{2}}d^{u}_{ob}e^{\frac{(1-i)}{\sqrt{2}}y_{\infty}}\Big] + \frac{\epsilon^{2}}{\epsilon^{u}}\Big[b^{u}_{o1} + \frac{(1-i)}{\sqrt{2}}d^{u}_{o1}e^{\frac{(1-i)}{\sqrt{2}}y_{\infty}}\Big] +$$

$$+ b^{u}_{1o} + d^{u}_{1o}e^{\frac{(1-i)}{\sqrt{2}}y_{\infty}}\frac{(1-i)}{\sqrt{2}} + \ldots$$

$$d^{u}_{ij} = 0$$

$$b^{u}_{oo} = b^{u}_{o1} = 0$$

$$iA^{u}_{oo} = b^{u}_{1o}$$

$$\begin{matrix} o \\ o \\ o \end{matrix}$$

The equations for the lower region may be obtained by interchanging the superscripted indices.

5.18 (vertical velocity)

$$\Phi_y^\beta \bigg|_{y = \pm \epsilon^\beta y_\infty} = \pm \Psi_x^\beta \bigg|_{y = y_\infty} ; \quad \beta = \left\{ \begin{matrix} \ell \\ u \end{matrix} \right\}$$

Case 1:

Upper Region:

$$- A_{oo}^u - \epsilon^\ell \left[A_{o1}^u + A_{oo}^u K_{10} \right] - \epsilon^u \left[A_{10}^u + A_{oo}^u K_{o1} - A_{oo}^u y_\infty \right] \cdots$$

$$= - i a_{oo}^u - i \epsilon^\ell \left[a_{o1}^u + a_{oo}^u K_{10} \right] - i \epsilon^u \left[a_{10}^u + b_{10}^u y_\infty + a_{oo}^u K_{o1} \right] \cdots$$

Therefore, $- i a_{oo}^u = - A_{oo}^u$,

$- i a_{o1}^u = - A_{o1}^u$, and,

$- i a_{10}^u = - A_{10}^u$, since

$- i b_{10}^u y_\infty = A_{oo}^u y_\infty$ (horizontal condition)

Lower Region:

$$+ A_{oo}^\ell + \epsilon^u \left[A_{o1}^\ell + A_{oo}^\ell K_{o1} \right] + \epsilon^\ell \left[A_{10}^\ell + A_{oo}^\ell K_{10} - A_{oo}^\ell y_\infty \right] \cdots$$

$$= i a_{oo}^\ell + \epsilon^u \left[i a_{o1}^\ell + i a_{oo}^\ell K_{o1} \right] + \epsilon^\ell \left[i a_{10}^\ell + b_{10}^\ell y_\infty i + i a_{oo}^\ell K_{1b} \right] \cdots$$

Therefore, $i a_{oo}^\ell = A_{oo}^\ell$,

$i a_{o1}^\ell = A_{o1}^\ell$, and,

$i a_{10}^\ell = A_{10}^\ell$, since,

$i b_{10}^\ell y_\infty = - A_{oo}^\ell y_\infty$ (horizontal condition)

The interfacial kinematic boundary conditions provide additional relationships linking the coefficients in the interface equation (eg. 5.1) with those in the stream function expansions. The continuity of velocity condition is expressed in the text as equations (5.19) and (5.20). The linearized interfacial kinematic condition allows us to equate the vertical velocity with the time derivative of (5.1).

Horizontal velocity

$$\left\{ \frac{1}{\epsilon^\ell} \Psi_y^\ell = \frac{1}{\epsilon^u} \Psi_y^u \right\}_{y=0} \qquad \text{(linearized)}$$

$$b_{10}^\ell - \frac{(1-i)}{\sqrt{2}} c_{10}^\ell = b_{10}^u - \frac{(1-i)}{\sqrt{2}} c_{10}^u$$

Vertical velocity

$$-\Psi_x^u \bigg|_{y=0} = \Psi_x^\ell \bigg|_{y=0} = F_t \qquad \text{(linearized)}$$

$$-i a_{oo}^u + \epsilon^\ell (-i a_{o1}^u - i K_{10} a_{oo}^u) + \epsilon^u (-i \Psi_{10}^{u}(0) - i K_{o1} a_{oo}^u) + \dots$$

$$= i a_{oo}^\ell + \epsilon^\ell (i \Psi_{10}^\ell(0) + i K_{10} a_{oo}^\ell) + \epsilon^u (i a_{o1}^\ell + i K_{o1} a_{oo}^\ell) + \dots$$

$$= -i \xi_{oo} - i \epsilon^\ell \xi_{10} - i \epsilon^u \xi_{o1} + \dots$$

$$-a_{oo}^u = a_{oo}^\ell = -\xi_{oo}$$

$$-(a_{o1}^u + K_{10} a_{oo}^u) = \Psi_{10}^\ell(0) + K_{10} a_{o0}^\ell = -\xi_{10}$$

$$-(\Psi_{10}^u(0) + K_{o1} a_{oo}^u) = a_{o1}^\ell + K_{o1} a_{oo}^\ell = -\xi_{o1}$$

Appendix B: Matching and Boundary
Condition Expansions for the Oil Spill Problem

Functional Forms in Stream Function Expansion (7.3 & 7.4)

$$\psi_{ij}^u = a_{ij}^u + b_{ij}^u y + c_{ij}^u \sinh\left[\frac{(1-i)}{\sqrt{2}}(y-d)\right] + d_{ij}^u \cosh\left[\frac{(1-i)}{\sqrt{2}}(y-d)\right]$$

$$\psi_{ij}^l = a_{ij}^l + b_{ij}^l y + c_{ij}^l e^{-\frac{(1-i)}{\sqrt{2}}y} + d_{ij}^l e^{\frac{(1-i)}{\sqrt{2}}y}$$

$$i = 0, 1; \quad j = 0, 1, 2, \cdots$$

Potential Function Expansions (Equations 7.5 and 7.6)

$$\Phi^u = 0$$

$$\Phi^l = Re\left\{\alpha e^{ikx - it}\left[A_{00}^l + \epsilon^l A_{10}^l + \epsilon^u A_{01}^l + \cdots\right]e^{ky}\right\} + O(\alpha^2)$$

The differences between this problem and the mono-layer problem spring chiefly from the finite depth of the upper layer. This leads to the use of the hyperbolic functional forms for the stream function expansion in the upper layer. The pertinent dynamic conditions are discussed in the text. The condition of zero tangential stress and zero elasticity at the oil-air interface leads to the requirement that $d_{ij}^u = 0$. This simplifies the kinematic conditions considerably. They are developed as follows.

Upper Surface Kinematic Condition   (7.18 a)

$$a_{oo}^u + b_{oo}^u d = \xi_{oo}^u$$

$$(a_{o1}^u + b_{o1}^u d) + K_{10}(a_{oo}^u + b_{oo}^u d) = \xi_{10}^u$$

$$(a_{10}^u + b_{10}^u d) + K_{01}(a_{oo}^u + b_{oo}^u d) = \xi_{o1}^u$$

or,

$$a_{o1}^u + b_{o1}^u d = \xi_{10}^u - K_{10}\xi_{oo}^u$$

$$a_{10}^u + b_{10}^u d = \xi_{o1}^u - K_{01}\xi_{oo}^u$$

Interfactial Kinematic Condition (7.19 a)

$$a_{oo}^u - c_{oo}^u \sinh\frac{(1-i)}{\sqrt{2}}d = -a_{oo}^l - c_{oo}^l = \xi_{oo}^i$$

$$a_{o1}^u - c_{o1}^u \sinh\frac{(1-i)}{\sqrt{2}}d + K_{10}\xi_{oo}^i = -a_{10}^l - c_{10}^l + K_{10}\xi_{oo}^i = \xi_{10}^i$$

$$a_{10}^u - c_{10}^u \sinh\frac{(1-i)}{\sqrt{2}}d + K_{01}\xi_{oo}^i = -a_{o1}^l - c_{o1}^l + K_{01}\xi_{oo}^i = \xi_{o1}^i$$

or,

$$a_{o1}^u - c_{o1}^u \sinh\frac{(1-i)}{\sqrt{2}}d = -a_{10}^l - c_{10}^l = \xi_{10}^i - K_{10}\xi_{oo}^i$$

$$a_{10}^u - c_{10}^u \sinh\frac{(1-i)}{\sqrt{2}}d = -a_{o1}^l - c_{o1}^l = \xi_{o1}^i - K_{01}\xi_{oo}^i$$

Expanding (7.21) in terms of the upper stream function, we find the following equations for the b_{ij}^u coefficients. This corresponds to the normal stress boundary condition at the air-oil interface. The

derivation is discussed in the text (pages 115-120.)

$$-\left.\frac{\partial u}{\partial t}\right|_{y=d} + \left.(\epsilon^u)^2 \frac{\partial^2 u}{\partial x^2}\right|_{y=d} + \frac{\partial^2 u}{\partial y^2} + \theta(\alpha^2) =$$

$$\varpi_g F_{S_x} + \left.2(\epsilon^u)^2 \frac{\partial^2 W}{\partial x \partial y}\right|_{y=0} - \frac{\varpi_\sigma^u}{q} F_{S_{xxx}}$$

$$i\, b_{oo}^u = 0$$

$$i\, b_{o1}^u = 0$$

$$i\, b_{1o}^u = i\left(\varpi_g \xi_{oo}^i + \frac{\varpi_\sigma^u}{q} \xi_{oo}^i\right)$$

These results may be coupled with those springing from the continuity of horizontal velocity condition at the oil-water interface to provide equations for the c_{ij}^u coefficients. The linearized horizontal velocity conditions are:

$$b_{oo}^u + \frac{(1-i)}{\sqrt{2}} \cosh \frac{(1-i)}{\sqrt{2}} d = 0$$

$$- \frac{(1-i)}{\sqrt{2}} c_{oo}^l = 0$$

$$b_{o1}^u + \frac{(1-i)}{\sqrt{2}} \cosh \frac{(1-i)}{\sqrt{2}} d = 0$$

$$- \frac{(1-i)}{\sqrt{2}} c_{o1}^l = 0$$

$$b_{1o}^u + \frac{(1-i)}{\sqrt{2}} c_{1o}^u \cosh \frac{(1-i)}{\sqrt{2}} d = b_{1o}^l - \frac{(1-i)}{\sqrt{2}} c_{1o}^l$$

Whence:

$$\xi^u_{00} - \xi^i_{00} = 0$$

$$\xi^u_{10} - \xi^i_{10} = a^i_{01} - a^u_{01} + K_{10}(\xi^u_{00} - \xi^i_{00}) = 0$$

$$\xi^u_{01} - \xi^i_{01} = b^u_{10} d + c^u_{10} \sinh \frac{(1-i)}{\sqrt{2}} d$$

The normal stress condition at the interface has the following expansion:

$$-i A^l_{00} + q w_g \xi^u_{00} + w_g (1-q) \xi^i_{00}$$
$$= -w^u_\sigma \xi^u_{00} - w^i_\sigma \xi^i_{00}$$

$$-i A^l_{01} + q w_g \xi^u_{01} - q a^u_{00} d + w_g (1-q) \xi^i_{01} =$$
$$-w^u_\sigma (\xi^u_{01} + 2 \xi^u_{00} K_{01}) - w^i_\sigma (\xi^i_{01} + 2 \xi^i_{00} K_{01})$$

$$-i A^l_{10} + (i A^l_{00} + a^l_{00}) y_{00} + q w_s \xi^u_{10} + w_g (1-q) \xi^i_{10} =$$
$$- w^u_\sigma (\xi^u_{10} + 2 \xi^u_{00} K_{10}) - w^i_\sigma (\xi^i_{10} + 2 \xi^i_{00} K_{10})$$

The matching conditions in the lower (water) layer are identical to those developed for the monolayer problem and the reader is referred to Appendix A and Sections 5 and 6 of the text.